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THE ILL-POSEDNESS OF A MIXED PROBLEM IN A CYLINDRICAL DOMAIN FOR MULTIDIMENSIONAL HYPERBOLIC-ELLIPTIC EQUATIONS IN CANCER MODELING

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Abstract. This paper investigates a mixed boundary value problem in a cylindrical domain for a class of multidimensional hyperbolic-elliptic equations arising in mathematical models of malignant tumor growth. In particular, axisymmetric approximations and local models of tumor spread in brain tissues lead to problem formulations in cylindrical geometry. Under assumptions, the problem leads to a class of multidimensional hyperbolic-elliptic equations, i.e., mixed-type equations whose properties may change between hyperbolic and elliptic in different subdomains or according to the parameters and coefficients. Idealized cylindrical geometry is used as a convenient framework for rigorous mathematical analysis. The main results establish the ambiguity of the solutions to the mixed problem and provide an explicit representation of its classical solutions. These results contribute to the analysis of multidimensional mixed-type equations in bounded domains and may support further computational investigations of tumor growth models.

Keywords: ill-posedness, mixed problem, hyperbolic-elliptic equations, cylindrical domain, glioma growth modeling.

INTRODUCTION

Mathematical modeling of the spatiotemporal evolution of malignant tumors is an important area of research in applied mathematics and biomedicine [1, 2, 3, 4]. The study of brain glioma is of particular interest: tumor cells diffusely invade surrounding brain tissue and do not form a clear boundary between healthy and diseased areas. In mathematical models, the spatial behavior of tumor cell populations is typically described by partial differential equations that consider mass conservation, migration flows, and growth-related reaction mechanisms [5, 6, 7, 8, 9]. The tumor microenvironment also plays a crucial role in disease progression, as it regulates the distribution of oxygen, nutrients, metabolites such as lactate, and other factors that influence both cell proliferation and directional migration. However, many mathematical models assume that these microenvironmental variables change on a faster time scale than the tumor itself and are therefore treated within a quasi-steady-state framework. This assumption leads to a coupled system in which tumor cell density is described by a hyperbolic equation, while the microenvironmental field is represented by an elliptic equation. Thus, even at the level of the basic biomedical formulation, it is natural to consider mixed models that combine hyperbolic and elliptic components. A wide range of studies are reviewed in [10, 11, 12].

For the analytical and computational study of tumor propagation, simplified geometric settings are often employed. Axisymmetric approximations and local models of tumor spread in brain tissue naturally led to formulations in cylindrical geometry. Although such an idealized geometry is not intended to reproduce the anatomical structure of the brain in detail, it provides a convenient mathematical framework for rigorous analysis of the model and for identifying qualitative effects that are relevant to the development of more realistic descriptions.

Under the above assumptions, the problem leads to a class of multidimensional hyperbolicelliptic equations, i.e., mixed-type equations whose properties may change between hyperbolic and elliptic in different subdomains or according to the parameters and coefficients. Thus, in [13], an overdetermined Cauchy problem is studied, and it is shown that the ill-posedness of the formulation requires the imposition of additional restrictions on the initial data. Their foundations were established in the classical works [14, 15, 16]. Nevertheless, for multidimensional mixed-type equations, particularly in bounded domains of complex geometry, a number of issues related to the correctness of the formulation of boundary value problems still remain unresolved.

A mixed problem in a cylindrical domain for multidimensional hyperbolicelliptic equations constitute a natural formulation when the transport dynamics of tumor cells are considered together with quasi-stationary microenvironmental fields. For multidimensional hyperbolic equations in spaces of generalized functions, the theory of mixed problems is already well established, including well-posedness results and solution representation [17, 18, 19], and elliptic boundary value problems are likewise grounded in a classical and developed theory. Nevertheless, for multidimensional hyperbolicelliptic equations, particularly in bounded domains such as cylindrical regions, the well-posedness of mixed boundary value problems remains largely unexplored. In this work, we prove the non-uniqueness of solutions and derive an explicit representation of classical solutions to the mixed problem for a multidimensional hyperbolicelliptic equation.

Materials and Methods

Let as in [19, 20] $\Omega_{\alpha\beta}$ be the finite region of the Euclidean space E_{m+1} of points $(x_1,\ldots,x_m,t)$, bounded at $t>0$ by the cylinder $\Gamma_\beta=\{(x,t):|x|=1\}$ and the plane $t=\beta>0$, and for $t<0$ the cylinder $\Gamma_\alpha=\{(x,t):|x|=1\}$ and the plane $t=\alpha<0$, where $|x|$ is the length of the vector $x=(x_1,\ldots,x_m)$, $m\geq 2$. Denote by Ω_β^+ and Ω_α^- parts of the domain $\Omega_{\alpha\beta}$, lying in the half-spaces $t>0$ and $t<0$; σ_β – the upper base of the domain Ω_β^+,a σ_α – the lower base of the domain Ω_α^-.

Let S be the common part of the boundaries of the regions Ω_β^+ and Ω_α^-, representing the set of $\{t=0, 0<|x|<1\}$ points from E_m.

In the domain of $\Omega_{\alpha\beta}$, consider multidimensional hyperbolo-elliptic equations

$$\begin{gathered}\Delta_x u-(sgnt)u_{tt}+\sum_{i=1}^{m}a_i(x,t)u_{x_i}+\\ +b(x,t)u_t+c(x,t)u=0,\end{gathered}\tag{1}$$

where Δ_x is the Laplace operator for variables $x_1,\ldots,x_m$.

Next, it is convenient for us to move from Cartesian coordinates $x_1,\ldots,x_m,t$ to spherical $r,\theta_1,\ldots,\theta_{m-1},t$, $r\geq 0$, $0\leq\theta_1<2\pi$, $0\leq\theta_i\leq\pi$, $i=\overline{2,m-1}$, $\theta=(\theta_1,\ldots,\theta_{m-1})$.

As a multidimensional mixed problem, consider the following problem:

Problem 1. *To find a solution of the equation (1) in the domain $\Omega_{\alpha\beta}$ at $t\neq 0$ from the class $C(\overline{\Omega}_{\alpha\beta})\cap C^1(\Omega_{\alpha\beta})\cap C^2(\Omega_\beta^+\cup\Omega_\alpha^-)$ satisfying the boundary conditions*

$$u\Big|_{\Gamma_\beta}=\psi_1(t,\theta),\tag{2}$$

$$u\Big|_{\Gamma_\alpha}=\psi_2(t,\theta),\quad u\Big|_{\sigma_\alpha}=\varphi(r,\theta),\tag{3}$$

with $\psi_1(0,\theta)=\psi_2(0,\theta)$, $\psi_2(\alpha,\theta)=\varphi(r,\theta)$.

Let $\{Y_{n,m}^k(\theta)\}$ be a system of linearly independent spherical functions of the order n, $1\leq k\leq k_n$, $(m-2)!n!k_n=(2n+m-3)!(2n+m-2)$, $W_2^l(S)$, $l=0,1,\ldots$ is Sobolev spaces [19].

There is a [21]:

Lemma 1. *Let $f(r,\theta)\in W_2^l(S)$. If $l\geq m-1$, then the series*

$$f(r,\theta)=\sum_{n=0}^{\infty}\sum_{k=1}^{k_n}f_n^k(r)Y_{n,m}^k(\theta),\tag{4}$$

and also the series obtained from it by differentiating the order $p\leq l-m+1$ converge absolutely and uniformly.

Lemma 2. [21]. *In order for $f(r,\theta)\in W_2^l(S)$ it is necessary and sufficient that the coefficients of the series (4) satisfy the inequalities*

$$|f_0^1(r)|\leq c_1,\quad \sum_{n=1}^{\infty}\sum_{k=1}^{k_n}n^{2l}|f_n^k(r)|^2\leq c_2,$$

$$c_1, c_2 = const.$$

Through $\widetilde{a}_{in}^k(r,t)$, $a_{in}^k(r,t)$, $\widetilde{b}_{in}^k(r,t)$, $\widetilde{c}_{in}^k(r,t)$, ρ_n^k, $\psi_{2n}^k(t)$, $\overline{\varphi}_n^k(r)$, denote the expansion coefficients of the series (4), respectively, of the functions $a_i(r,\theta,t)\rho$, $a_i\frac{x_i}{r}\rho$, $i=1,\ldots,m$, $b(r,\theta,t)\rho$, $c(r,\theta,t)\rho$, $\rho(\theta)$, $\psi_2(t,\theta)$, $\varphi(r,\theta)$.

Let $a(r,\theta,t)$, $b(r,\theta,t)$, $c(r,\theta,t) \in W_2^l(\Omega_\beta^+ \cup \Omega_\alpha^-)$, $i=1,\ldots,m$, $l \geq m+1$.

Then the following is true

Theorem 1. *If $\psi_1(t,\theta) \in W_2^l(\Gamma_\beta)$, $\psi_2(t,\theta) \in W_2^l(\Gamma_\alpha)$, $\varphi(r,\theta) \in W_2^l(\sigma_\alpha)$, $l > \frac{3m}{2}$, then Problem 1 is solvable and ambiguous.*

Note that this theorem for the multidimensional Lavrentiev-Bitsadze equation is obtained in [20].

Proof. In spherical coordinates, the equation (1) in the domain Ω_α^- has the form [21, 22]

$$Lu \equiv u_{rr} + \frac{m-1}{r}u_r - \frac{1}{r^2}\delta u + u_{tt} + \sum_{i=1}^{m} a_i(r,\theta,t)u_{x_i} + b(r,\theta,t)u_t + c(r,\theta,t)u = 0, \tag{5}$$

where is

$$\delta \equiv -\sum_{j=1}^{m-1} \frac{1}{g_j \sin^{m-j-1}\theta_j}\frac{\partial}{\partial\theta_j}\left(\sin^{m-j-1}\theta_j \frac{\partial}{\partial\theta_j}\right),$$

$$g_1 = 1,\ g_j = (\sin\theta_1 \ldots \sin\theta_{j-1})^2,\ j > 1.$$

It is known [19, 21, 22] that the spectrum of the operator δ consists of eigenvalues $\lambda_n = n(n+m-2)$, $n = 0,1,\ldots$, each of which corresponds to k_n orthonormal functions $Y_{n,m}^k(\theta)$.

The desired solution to Problem 1 in the domain Ω_α^- will be searched in the form

$$u(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{k=1}^{k_n} \overline{u}_n^k(r,t)Y_{n,m}^k(\theta), \tag{6}$$

where $\overline{u}_n^k(r,t)$ are the functions to be defined.

Substituting (6) into (5), then multiplying the resulting expression by $\rho(\theta) \neq 0$, and integrating over the unit sphere H into E_m for $\overline{u}_n^k$, we obtain the equation [22]

$$\rho_0^1\overline{u}_{0rr}^1 + \rho_0^1\overline{u}_{0tt}^1 + \left(\frac{m-1}{r}\rho_0^1 + \sum_{i=1}^{m} a_{i0}^1\right)\overline{u}_{0r}^1 + \widetilde{b}_0^1\overline{u}_{0t}^1 + \widetilde{c}_0^1\overline{u}_0^1 + \sum_{n=1}^{\infty}\sum_{k=1}^{k_n}\Big\{\rho_n^k\overline{u}_{nrr}^k + \rho_n^k\overline{u}_{ntt}^k + \left(\frac{m-1}{r}\rho_n^k + \sum_{i=1}^{m} a_{in}^k\right)\overline{u}_{nr}^k + \widetilde{b}_n^k\overline{u}_{nt}^k + [\widetilde{c}_n^k - \lambda_n\frac{\rho_n^k}{r^2} + \sum_{i=1}^{m}(\widetilde{a}_{in-1}^k - na_n^k)]\overline{u}_n^k\Big\} = 0. \tag{7}$$

Now consider an infinite system of differential equations

$$\rho_0^1\overline{u}_{0rr}^1 + \rho_0^1\overline{u}_{0tt}^1 + \frac{m-1}{r}\rho_0^1\overline{u}_{0r}^1 = 0, \tag{8}$$

$$\rho_1^k\overline{u}_{1rr}^k + \rho_1^k\overline{u}_{1tt}^k + \frac{m-1}{r}\rho_1^k\overline{u}_{1r}^k - \frac{\lambda_1}{r^2}\rho_1^k\overline{u}_1^k = -\frac{1}{k_1}\left(\sum_{i=1}^{m} a_{i0}^1\overline{u}_{0r}^1 + \widetilde{b}_0^1\overline{u}_{0t}^1 + \widetilde{c}_0^1\overline{u}_0^1\right),\quad n=1,\ k=\overline{1,k_1}, \tag{9}$$

$$\rho_n^k\overline{u}_{nrr}^k + \rho_n^k\overline{u}_{ntt}^k + \frac{m-1}{r}\rho_n^k\overline{u}_{nr}^k - \frac{\lambda_n}{r^2}\rho_n^k\overline{u}_n^k = -\frac{1}{k_n}\sum_{k=1}^{k_{n-1}}\Big\{\sum_{i=1}^{m} a_{in-1}^k\overline{u}_{n-1r}^k + \widetilde{b}_{n-1}^k\overline{u}_{n-1t}^k + [\widetilde{c}_{n-1}^k + \sum_{i=1}^{m}\left(\widetilde{a}_{in-1}^k - (n-1)a_{in-1}^k\right)]\overline{u}_{n-1}^k\Big\},\quad k=\overline{1,k_1},\ n=2,3,\ldots. \tag{10}$$

Summing up the equation (9) from 1 to k_1, and the equation (10) from 1 to k_n, and then adding the resulting expressions together with (8) we come to the equation (7).

It follows that if $\{\overline{u}_n^k\}$, $k=\overline{1,k_n}$, $n=0,1,\ldots$ is the solution of the system (8) – (10), then it is also the solution the equations (7).

Note that each equation of the (8) –(10) system can be represented as [23]

$$\overline{u}_{nrr}^k + \overline{u}_{ntt}^k + \frac{m-1}{r}\overline{u}_{nr}^k - \frac{\lambda_n}{r^2}\overline{u}_n^k = f_n^k(r,t), \tag{11}$$

where $\overline{f}_n^k(r,t)$ is determined from the previous equations of this system, with $f_0^1(r,t) \equiv 0$.

At the same time, from the boundary condition (3) by virtue of (6), taking into account Lemma 1, we obtain

$$\overline{u}_n^k(1,t)=\psi_n^k(t),\ \ \overline{u}_n^k(r,\alpha)=\overline{\varphi}_n^k(r),\quad k=\overline{1,k_n},\ \ n=0,1,\ldots. \tag{12}$$

In (11), (12), replacing $\overline{v}_n^k(r,t)=\overline{u}_n^k(r,t)-\psi_n^k(t)$, we get

$$\overline{v}_{nrr}^k+\frac{m-1}{r}\overline{v}_{nr}^k-\frac{\lambda_n}{r^2}\overline{v}_n^k+\overline{v}_{ntt}^k=\overline{f}_n^k(r,t), \tag{13}$$

$$\begin{gathered}\overline{v}_n^k(1,t)=0,\ \ \overline{v}_n^k(r,\alpha)=\varphi_n^k(r),\\ k=\overline{1,k_n},\ \ n=0,1,\ldots,\\ \overline{f}_n^k(r,t)=f_n^k(r,t)+\frac{\lambda_n}{r^2}\psi_{2n}^k(t)-\psi_{2ntt}^k,\\ \varphi_n^k(r)=\overline{\varphi}_n^k(r)-\psi_n^k(\alpha).\end{gathered} \tag{14}$$

Replacing $\overline{v}_n^k(r,t)=r^{(1-m)/2}v_n^k(r,t)$, we reduce the problem (13), (14) to the following problem [20]

$$Lv_n^k=v_{nrr}^k+\frac{\overline{\lambda}_n}{r^2}v_n^k+v_{ntt}^k=\widetilde{f}_n^k(r,t), \tag{15}$$

$$\begin{gathered}v_n^k(1,t)=0,\ \ v_n^k(r,\alpha)=\widetilde{\varphi}_n^k(r),\\ k=\overline{1,k_n},\ \ n=0,1,\ldots,\\ \overline{\lambda}_n=\frac{[(m-1)(3-m)-4\lambda_n]}{2},\\ \widetilde{f}_n^k(r,t)=r^{(m-1)/2}\overline{f}_n^k(r,t),\\ \widetilde{\varphi}_n^k(r)=r^{(m-1)/2}\varphi_n^k(r).\end{gathered} \tag{16}$$

We seek a solution to problem (15), (16) in the form $v_n^k(r,t)=v_{1n}^k(r,t)+v_{2n}^k(r,t)$, where $v_{1n}^k(r,t)$ – solution of the problem

$$Lv_{1n}^k=\widetilde{f}_n^k(r,t),\ v_{1n}^k(1,t)=0,\ v_{1n}^k(r,\alpha)=0, \tag{17}$$

and $v_{2n}^k(r,t)$ – solving the problem

$$Lv_{2n}^k=0,\ \ v_{2n}^k(1,t)=0,\ \ v_{2n}^k(r,\alpha)=\widetilde{\varphi}_n^k(r). \tag{18}$$

The solution of the problem mentioned above is sought in the following form:

$$v_n^k(r,t)=\sum_{s=1}^{\infty}R_s(r)T_s(t). \tag{19}$$

Also, let

$$\begin{gathered}\widetilde{f}_n^k(r,t)=\sum_{s=1}^{\infty}a_{s,n}^k(t)R_s(r),\\ \widetilde{\varphi}_{2n}^k(r)=\sum_{s=1}^{\infty}b_{s,n}^kR_s(r).\end{gathered} \tag{20}$$

Substituting (19) into (17), taking into account (20), we get

$$\begin{gathered}R_{srr}+\left(\frac{\overline{\lambda}_n}{r^2}+\mu\right)R_s=0,\\ 0<r<1,\ \ R_s(1)=0,\ \ |R_s(0)|<\infty,\end{gathered} \tag{21}$$

$$T_{stt}-\mu T_s(t)=a_{s,n}^k(t),\ \ \alpha<t<0 \tag{22}$$

$$T_s(\alpha)=0. \tag{23}$$

A limited solution to the (20) problem is [24]

$$R_s(r)=\sqrt{r}J_\nu(\mu_{s,n}r),\ \ \mu=\mu_{s,n}^2. \tag{24}$$

$\nu=n+(m-2)/2$, $\mu_{s,n}$ – zeros of Bessel functions of the first kind $J_\nu(z)$, $\mu=\mu_{s,n}^2$.

The general solution of the equation (22) is represented as [24]

$$\begin{gathered}T_{s,n}(t)=c_{1s}\cosh\mu_{s,n}t+c_{2s}\sinh\mu_{s,n}t-\\ -\frac{\cosh\mu_{s,n}t}{\mu_{s,n}}\int_t^0 a_{s,n}^k(\xi)\sinh\mu_{s,n}\xi d\xi+\\ -\frac{\sinh\mu_{s,n}t}{\mu_{s,n}}\int_t^0 a_{s,n}^k(\xi)\cosh\mu_{s,n}\xi d\xi,\end{gathered}$$

c_{1s}, c_{2s} – arbitrary constants, satisfying the condition (23), we will have

$$\begin{gathered}\mu_{s,n}T_{s,n}(t)=c_{1s}\mu_{s,n}[\cosh\mu_{s,n}t-\\ -(\coth\mu_{s,n}\alpha)\sinh\mu_{s,n}t]+\\ +\left[(\coth\mu_{s,n}\alpha)\int_\alpha^0 a_{s,n}^k(\xi)(\sinh\mu_{s,n}\xi)d\xi-\right.\\ \left.-\int_\alpha^0 a_{s,n}^k(\xi)(\cosh\mu_{s,n}\xi)d\xi\right]\sinh\mu_{s,n}t-\\ -(\cosh\mu_{s,n}t)\int_t^0 a_{s,n}^k(\xi)\sinh\mu_{s,n}\xi d\xi+\\ +(\sinh\mu_{s,n}t)\int_t^0 a_{s,n}^k(\xi)\cosh\mu_{s,n}\xi d\xi.\end{gathered} \tag{25}$$

Substituting (24) into (20), we get

$$r^{-\frac{1}{2}}\widetilde{f}_n^k(r,t)=\sum_{s=1}^{\infty}a_{s,n}^k(t)J_\nu(\mu_{s,n}r),$$
$$r^{-\frac{1}{2}}\widetilde{\varphi}_{3n}^k(r)=\sum_{s=1}^{\infty}b_{s,n}^kJ_\nu(\mu_{s,n}r),\ 0<r<1. \tag{26}$$

Series (26) – expansions into Fourier-Bessel series [25] if

$$a_{s,n}^k(t)=2[J_{\nu+1}(\mu_{s,n})]^{-2}\cdot \int_0^1\sqrt{\xi}\widetilde{f}_n^k(\xi,t)J_\nu(\mu_{s,n}\xi)d\xi, \tag{27}$$

$$b_{s,n}^k=2[J_{\nu+1}(\mu_{s,n})]^{-2}\cdot \int_0^1\sqrt{\xi}\widetilde{\varphi}_n^k(\xi)J_\nu(\mu_{s,n}\xi)d\xi, \tag{28}$$

where $\mu_{s,n}$, $s=1,2,\dots$ are the positive zeros of the Bessel functions $J_\nu(z)$, arranged in ascending order of their magnitude.

From (24) and (25), the solution to problem (17) is obtained in the following form:

$$v_{1n}^k(r,t)=\sum_{s=1}^{\infty}\sqrt{r}T_{s,n}(t)J_\nu(\mu_{s,n}r), \tag{29}$$

where $T_{s,n}(t)$ – are determined from (25), and $a_{s,n}^k(t)$ – from (27).

Next, substituting (19) into (18), taking into account (20) we will have

$$V_{stt}-\mu_{s,n}^2V_s=0,\ \ \alpha<t<0, \tag{30}$$

$$V_s(\alpha)=b_{s,n}^k. \tag{31}$$

The general solution of the equation (30) has the form

$$V_{s,n}(t)=c'_{1s}\cosh\mu_{s,n}t+c'_{2s}\sinh\mu_{s,n}t,$$

where c'_{1s}, c'_{2s} are arbitrary constants, satisfying which condition (31) we get

$$V_{s,n}(t)=c'_{1s}[\cosh\mu_{s,n}t-(\coth\mu_{s,n}\alpha)\sinh\mu_{s,n}t]+ +\frac{b_{s,n}^k\sinh\mu_{s,n}t}{\sinh\mu_{s,n}\alpha} \tag{32}$$

From (24), (32) we get the solution to the problem (18) by the formula

$$v_{2n}^k(r,t)=\sum_{s=1}^{\infty}\sqrt{r}V_{s,n}(t)J_\nu(\mu_{s,n}r), \tag{33}$$

where $V_{s,n}(t)$ are from (32), and $b_{s,n}^k$ are from (28).

Therefore, first solving the problem (8), (11) $(n=0)$, and then (9), (11), etc., we find sequentially all $v_n^k(r,t)=v_{1n}^k(r,t)+v_{2n}^k(r,t)$, where $v_{1n}^k(r,t)$, $v_{2n}^k(r,t)$ are defined from (29), (33), $k=1,k_n$, $n=0,1,\dots$.

So in the area of Ω_α^- there is a

$$\int_H\rho(\theta)LudH=0. \tag{34}$$

Let $f(r,\theta,t)=R(r)\rho(\theta)T(t)$, and $R(r)\in V_0$, V_0 – is dense in $L_2((0,1))$, $\rho(\theta)\in C^\infty(H)$ – dense in $L_2(H)$, a $T(t)\in V_1$, V_1 – dense in $L_2((\alpha,0))$. Then $f(r,\theta,t)\in V$, $V=V_0\otimes H\otimes V_1$ – is dense $L_2((\Omega_\alpha^-))$ [26].

It follows from this and from (34) that

$$\int_{\Omega_\alpha^-}f(r,\theta,t)Lud\Omega_\alpha^-=0$$

and

$$Lu=0,\quad\forall(r,\theta,t)\in\Omega_\alpha^-.$$

Thus, the boundary value problem for the equation (5) with data $u\big|_{\Gamma_\alpha}=\psi_2(t,\theta)$, $u\big|_{\sigma_\alpha}=\varphi_2(r,\theta)$ in the domain Ω_α^- has countless solutions of the form

$$u(r,\theta,t)=\sum_{n=1}^{\infty}\sum_{k=1}^{k_n}\{\psi_n^k(t)+ +r^{(1-m)/2}[v_{1n}^k(r,t)+v_{2n}^k(r,t)]\}Y_{n,m}^k(\theta), \tag{35}$$

where $v_{1n}^k(r,t)$,$v_{2n}^k(r,t)$ are from (29), (33).

Using the formula [25] $2J'_\nu(z)=J_{\nu-1}(z)-J_{\nu+1}(z)$, estimates [21, 25]

$$|J_\nu(z)|\le\frac{1}{\Gamma(1+\nu)}\left(\frac{z}{2}\right)^\nu,\ \ |k_n|\le c_1n^{m-2},$$

$$\left|\frac{\partial^q}{\partial\theta_j^q}Y_{n,m}^k(\theta)\right|\le c_2n^{\frac{m}{2}-1+q},\ \ c_1,c_2=const,$$

$$j=\overline{1,m-1},\quad q=0,1,\ldots,$$

$\Gamma(z)$ is the gamma function, lemmas, constraints on the coefficients of the equation (5) and on the given functions $\psi_2(t,\theta)$, $\varphi(r,\theta)$, as well as embedding theorem [26], as in [27], it can be shown that the obtained solution (35) belongs to the class $C(\overline{D_\alpha^-})\cap C^2(D_\alpha)$.

Next, from (29), (33) and (35) at $t\to-0$ it will have

$$u(r,\theta,0)=\tau(r,\theta)=\sum_{n=1}^{\infty}\sum_{k=1}^{k_n}\sum_{s=1}^{\infty}\{\psi_{2n}^k(0)+$$

$$+r^{\frac{(2-m)}{2}}(c_{1s}+c'_{1s})\}J_{n+\frac{(m-2)}{2}}(\mu_{s,n}r)Y_{n,m}^k(\theta),$$

$$u_t(r,\theta,0)=\nu(r,\theta)=\sum_{n=1}^{\infty}\sum_{k=1}^{k_n}\sum_{s=1}^{\infty}\Big\{\psi_{2nt}^k(0)+$$
$$+r^{\frac{(2-m)}{2}}\Big[-(c_{1s}+c'_{1s})\mu_{s,n}\coth\mu_{s,n}\alpha+$$

$$+(\coth\mu_{s,n}\alpha)\int_{\alpha}^{0}a_{s,n}^k(\xi)(\sinh\mu_{s,n}\xi)d\xi-$$
$$-\int_{\alpha}^{0}a_{s,n}^k(\xi)(\cosh\mu_{s,n}\xi)d\xi+$$

$$+\frac{\mu_{s,n}b_{s,n}^k}{\cosh\mu_{s,n}\alpha}\Big]\Big\}J_{n+\frac{(m-2)}{2}}(\mu_{s,n}r)Y_{n,m}^k(\theta),\tag{36}$$

in this case, $\tau(r,\theta)\in C(\overline{S})\cap C^2(S), \nu(r,\theta)\in W_2^l(S)$, $l>\frac{3m}{2}$.

Now we will study Problem 1 in the domain of Ω_β^+, which, by virtue of (2) and (36), reduces to a mixed problem for a multidimensional hyperbolic equation

$$u_{rr}+\frac{(m-1)}{r}u_r-\frac{1}{r^2}\delta u-u_{tt}+$$
$$+\sum_{i=1}^{m}a_i(r,\theta,t)u_{x_i}+b(r,\theta,t)u_t+c(r,\theta,t)u=0\tag{37}$$

with the condition

$$u\Big|_S=\tau(r,\theta),\quad u_t\Big|_S=\nu(r,\theta),\quad u\Big|_{\Gamma_\beta}=\psi_1(t,\theta).\tag{38}$$

The following is shown in [19, 27]

Theorem 2. *The problem* (37), (38) *is uniquely solvable in the class* $C(\overline{\Omega}_\beta^+)\cap C^2(\Omega_\beta^+)$.

From the representation of the functions $\tau(r,\theta)$, $\nu(r,\theta)$ in the form of (36) [19], and also from Theorem 2 it follows that Problem 1 has countless classical solutions.

Theorem 1 has been proved.

Since in [19, 27] an explicit form of solutions to the problem (37), (38) is obtained, it is possible to write an explicit representation of the solution for Problem 1.

Results and Discussion

The main theoretical result shows that problem admits ambiguous classical solutions. This ambiguity is due to the mixed hyperbolic-elliptic structure of the operator and the influence of cylindrical geometry. The result obtained allow us to identify the mathematical mechanisms underlying the non-uniqueness of solutions in such models and provide a theoretical basis for further research into direct and inverse problems in mathematical oncology. The explicit solutions obtained in this work can also serve as benchmark test cases for numerical methods developed for mixed-type equations and for further theoretical studies of the well-posedness of problem formulations in a more general setting. We must understand that mathematical modeling provides significant insight into glioma growth, but no one can claim yet that it provides a complete picture of the disease due to the multifactorial complexity of the disease itself. A comprehensive literature review can be found not only in [1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 20, 28] but also in the articles they cited.

Conclusions

In this paper, we proved the ambiguity of solutions and obtained an explicit from for for classical solutions of a mixed problem for multidimensional hyperbolic-elliptic equations. These results are relevant both for the theory of mixed-type equations and for applications where conditions or regularization may be required to ensure well-posedness. In the study of direct and inverse tumor growth problems, the sensitivity of the coefficients (conditions) leads to the ill-posedness problem. Therefore, studying the well-posedness of multidimensional hyperbolic-elliptic equations is of interest both from the perspective of the theory of partial differential equations and in the context of mathematical modeling of tumor growth processes.

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