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THREE FRACTIONAL INEQUALITIES BASED ON A SINGLE PARAMETER IN THE CONTEXT OF *AG*-MULTIPLICATIVE *h*-CONVEX FUNCTIONS

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Abstract. This article introduces novel three parameterized fractional inequalities based on a one parameter ν derived from functions that are *AG*-multiplicative differentiable, as characterized by *AG*-multiplicatively Riemann-Liouville operators (classical, middle starting point, and middle ending point). By employing the multiplicative absolute value and supposing the function is *AG*-multiplicative *h*-convex, we can develop three identities and utilize them to derive a series of inequalities for *AG*-multiplicatively *h*-convex mappings. We additionally present these findings for *AG*-multiplicative *P*-functions convex mapping and *AG*-multiplicative *s*-convex functions. The last section delineates specific instances, including trapezoid, midpoint, Bullen, Milne, Simpson and corrected Simpson *AG*-multiplicative fractional inequalities. We further illustrate that certain results reported here improve established results, while others represent generalizations.

Keywords: Parameterized inequality, *AG*-multiplicative fractional integrals, Multiplicative absolute value, *AG*-multiplicative *h*-convex functions.

INTRODUCTION

Newton and Leibniz's primary work has been associated with the concept of *AG*-multiplicative calculus. Grossman and Katz further developed this concept through an investigation and evaluation of the original non-Newtonian systems[1]. Bashirov et al. [2] introduced the concepts of integral and *AG*-multiplicative derivatives, along with the methodologies for -*AG*-multiplicative integrals and derivatives. The procedures for *AG*-multiplicative integrals and derivatives were also demonstrated.

Definition 1. For a function $G : I^{\circ} \subseteq \mathbb{R} \to \mathbb{R}^{+}$, the *AG*-multiplicative integral of f, represented by $\int_{\sigma_1}^{\sigma_2}(f(\tau))^{d\tau}$, is defined as

$$\int_{\sigma_1}^{\sigma_2}(f(\tau))^{d\tau} = \exp\left\{\int_{\sigma_1}^{\sigma_2}\ln(f(\tau))d\tau\right\}. \quad (1)$$

The following theorem addresses various properties of *AG*-multiplicative integrals.

Theorem 2. *If f and g are positive and Riemann integrable on the interval $[\sigma_1,\sigma_2] \subset I^{\circ}$, $p \in \mathbb{R}$, then f and g are $*$-integrable on $[\sigma_1,\sigma_2]$ and*

- $\int_{\sigma_1}^{\sigma_2}((f(\tau))^p)^{d\tau} = \left(\int_{\sigma_1}^{\sigma_2}(f(\tau))^{d\tau}\right)^p.$
- $\int_{\sigma_1}^{\sigma_2}(f(\tau)\cdot g(\tau))^{d\tau}$
$= \int_{\sigma_1}^{\sigma_2}(f(\tau))^{d\tau}\cdot\int_{\sigma_1}^{\sigma_2}(g(\tau))^{d\tau}.$
- $\int_{\sigma_1}^{\sigma_2}\left(\frac{f(\tau)}{g(\tau)}\right)^{d\tau} = \dfrac{\int_{\sigma_1}^{\sigma_2}(f(\tau))^{d\tau}}{\int_{\sigma_1}^{\sigma_2}(g(\tau))^{d\tau}}.$
- $\int_{\sigma_1}^{c}(f(\tau))^{d\tau}\cdot\int_{c}^{\sigma_2}(f(\tau))^{d\tau}$
$= \int_{\sigma_1}^{\sigma_2}(f(\tau)); \quad \sigma_1 \le c \le \sigma_2.$
- $\int_{\sigma_1}^{\sigma_1}(f(\tau))^{d\tau} = 1$
- $\int_{\sigma_1}^{\sigma_2}(f(\tau))^{d\tau} = \left(\int_{\sigma_2}^{\sigma_1}(f(\tau))^{d\tau}\right)^{-1}.$

Definition 3. Given a function $f : I^{\circ} \subseteq \mathbb{R} \to \mathbb{R}^{+}$, the *AG*-multiplicative derivative (or $*$derivative) of f, denoted by f^*, is given by

$$f^*(\tau)=\lim_{z\to 0}\left(\frac{f(\tau+z)}{f(\tau)}\right)^{\frac{1}{z}}.$$

Remark 4. *For a positive differentiable function f, a corresponding AG-multiplicative derivative f^* exists, and the relationship between f^* and f' can be expressed using the following formulas:*

$$\begin{aligned} &f^*(\tau)=\exp\{(\ln\circ f)'(\tau)\},\\ \textit{or}\qquad &\\ &\ln\circ f^*=(\ln\circ f)'. \end{aligned} \tag{2}$$

The following theorem presents several properties of AG-multiplicative derivatives.

Theorem 5. *Let f and g be positive $*$differentiable functions. If k is an arbitrary positive constant, then functions kf, $f\cdot g$, $f+g$, $\frac{f}{g}$, f^g and $f\circ g$ are $*$differentiable and*

- $(kf)^*(\tau)=f^*(\tau)$.
- $(f\cdot g)^*(\tau)=f^*(\tau)g^*(\tau)$.
- $(f+g)^*(\tau)$ $$=f^*(\tau)^{\dfrac{f(\tau)}{f(\tau)+g(\tau)}}\cdot g^*(\tau)^{\dfrac{g(\tau)}{f(\tau)+g(\tau)}}.$$
- $\left(\dfrac{f}{g}\right)^*(\tau)=\dfrac{f^*(\tau)}{g^*(\tau)}$.
- $(f^g)^*(\tau)=f^*(\tau)^{g(\tau)}\cdot f(\tau)^{g'(\tau)}$.
- $(f\circ g)^*(\tau)=f^*(g(\tau))^{g'(\tau)}$.

Theorem 6. *[2, Theorem 6] (AG-Multiplicative integration by parts) Let f be a positive, AG-multiplicatively differentiable function on I° and $g:I^\circ\to\mathbb{R}$ differentiable and $[\sigma_1,\sigma_2]\subset I^\circ$, then the function $(f^*)^g$ is integrable and we have*

$$\begin{aligned}&\int_{\sigma_1}^{\sigma_2}\left((f^*(\tau))^{g(\tau)}\right)^{d\tau}\\ &=\frac{(f(\sigma_2))^{g(\sigma_2)}}{(f(\sigma_1))^{g(\sigma_1)}}\cdot\frac{1}{\displaystyle\int_{\sigma_1}^{\sigma_2}\left((f(\tau))^{g'(\tau)}\right)^{d\tau}}\end{aligned}\tag{3}$$

The next step involves the definition of AG-multiplicative h-convexity [3].

Definition 7. Let $h:[0,1]\to\mathbb{R}$ be a positive function. We say that the function $f:I^\circ\to\mathbb{R}^+$ is AG-multiplicatively h-convex ($*h$-convex) if for all $z,y\in[\sigma_1,\sigma_2]\subset I^\circ$ and $\tau\in[0,1]$ we have

$$f(\tau z+(1-\tau)y)\le[f(z)]^{h(\tau)}\,[f(y)]^{h(1-\tau)}. \tag{4}$$

If inequality (4) is reversed, then f is said to be AG-multiplicatively h-concave ($*h$-concave).

Remark 8. *The following relations can be derived from the definition that was presented earlier:*

- *If f and g are two AG-multiplicatively h-convex functions, then their product $f\cdot g$ is additionally a AG-multiplicatively h-convex function.*
- *If f is a AG-multiplicatively h-convex function, then it follows that $\frac{1}{f}$ is a AG-multiplicatively h-concave function.*
- *If g is a AG-multiplicatively h-concave function, then $\frac{1}{g}$ is a AG-multiplicatively h-convex function.*
- *In the case when f is a AG-multiplicatively h-convex function and g is a AG-multiplicatively h-concave function, the quotient $\frac{f}{g}$ will also be a AG-multiplicatively h-convex function.*
- *If f and g are two AG-multiplicatively h-convex functions, the quotient $\frac{f}{g}$ is not necessarily a AG-multiplicatively h-convex function. For example, let $g=f\cdot f$, where f is a AG-multiplicatively h-convex function. This results in $\frac{f}{g}=\frac{1}{f}$, which is a AG-multiplicatively h-concave function.*

The next discussion will cover various types of AG-multiplicative convexity.

1. The AG-multiplicatively $*s$-convex functions are obtained by setting $h(t)=t^s$, $s\in(0,1]$ in (4) [3].

$$f(\tau z+(1-\tau)x)\le[f(z)]^{\tau^s}\,[f(x)]^{(1-\tau)^s}.$$

2. By setting $h(t) = 1$ in (4), we derive *AG*-multiplicatively $*P$-functions [3].

$$f(\tau z + (1-\tau)x) \leq f(z) \cdot f(x).$$

3. By replacing $h(t) = t$ in (4), we derive the concept of *AG*-multiplicatively convex functions [4].

$$f(\tau z + (1-\tau)x) \leq [f(z)]^{\tau} \; [f(x)]^{1-\tau}.$$

Remark 9. *Since the function* $\ln(.)$ *is concave, for* $f(z), f(\tau) > 0$ *and* $\tau \in [0,1]$ *we obtain*

$$\begin{aligned} &\tau \ln f(z) + (1-\tau) \ln f(x) \\ &\leq \ln(\tau f(z) + (1-\tau) f(x)), \end{aligned}$$

therefore

$$[f(z)]^{\tau} \cdot [f(x)]^{1-\tau} \leq \tau f(z) + (1-\tau) f(x). \tag{5}$$

This signifies that a AG-multiplicatively convex function has convexity, although the converse is not necessarily valid.

In [5], the concept of a *B*-function was introduced as follows:

Definition 10. Let $H : [0,\infty) \to \mathbb{R}$ be a non-negative function. H is called a *B*-function if

$$H(t-\sigma_1) + H(\sigma_2 - t) \leq 2H\left(\frac{\sigma_1+\sigma_2}{2}\right) \tag{6}$$

where $\sigma_1 < t < \sigma_2$ with $\sigma_1, \sigma_2 \in [0,\infty)$.

In particular, taking $H = h$, $\sigma_1 = 0$ and $\sigma_2 = 1$ in (6), we obtain the inequality

$$h(t) + h(1-t) \leq 2h\left(\frac{1}{2}\right). \tag{7}$$

Examples of such a function h satisfying the inequality (7) can be provided by $h_1(t) = 1$, $h_2(t) = t$ and $h_3(t) = t^s$ with $s \in (0,1]$.

Definition 11. [6, p 44] The multiplicative absolute value is represented by $|.|_*$ in and is defined as follows:

$$|x|_* = \begin{cases} x & \text{if } x \geq 1 \\ \\ \dfrac{1}{x} & \text{if } 0 < x < 1. \end{cases}$$

Remark 12. *According to the previous Definition 11, we may conclude as follows:*

- *The absolute value* $|c-d|$ *for real numbers c and d is analogous to the multiplicative absolute value* $\left|\frac{z}{x}\right|_*$ *in the context of positive real numbers z and x.*

- *Using the notation* $\left|\frac{z}{x}\right|_*$ *indicates that* $\frac{z}{x} > 1$ *or* $\frac{z}{x} < 1$, *which corresponds to the conditions* $z > x$ *or* $z < x$, *respectively.*

- *In AG-multiplicative calculus, the notation* $\left|\frac{z}{x}\right|$ *does not have significance for positive real numbers z and x.*

Various innovative properties of the multiplicative absolute presented below:

Lemma 13. • *For the real number B, we have*

$$|\exp\{B\}|_* = \exp|B|. \tag{8}$$

- *If* $k \in \mathbb{R}$ *and* $z, x > 0$, *then the following are true.*

I-1 $|z|_* \geq 1$ *and* $|1|_* = 1$.

I-2 $\left|\frac{1}{x}\right|_* = |x|_*$ *and* $\frac{1}{|x|_*} = (|x|_*)^{-1}$.

I-3 $\left|\frac{z}{x}\right|_* = \left|\frac{x}{z}\right|_*$.

I-4 $\left|z^k\right|_* = (|z|_*)^{|k|}$.

I-5 $|z \times x|_* \leq |z|_* \times |x|_*$.

I-6 $\left|\frac{z}{x}\right|_* \leq |z|_* \times |x|_*$.

- *If* ψ *is a positive, AG-multiplicative function on* I° *with* $[\sigma_1, \sigma_2] \subset I^\circ$, *the following equality is valid:*

$$|\ln \circ \psi| = \ln \circ |\psi|_*. \tag{9}$$

The left-sided and right-sided Riemann-Liouville fractional integrals of order $\alpha > 0$ are defined as follows:

$$\mathscr{I}^{\alpha}_{\sigma_1+}f(x) = \frac{1}{\Gamma(\alpha)}\int_{\sigma_1}^{x}(x-t)^{\alpha-1}f(t)dt, \quad \sigma_1 < x,$$

$$\mathscr{I}^{\alpha}_{\sigma_2-}f(x) = \frac{1}{\Gamma(\alpha)}\int_{x}^{\sigma_1}(t-x)^{\alpha-1}f(t)dt, \quad x < \sigma_2.$$

Remark 14. *The operators* $\mathscr{I}^{\alpha}_{\left(\frac{\sigma_1+\sigma_2}{2}\right)^+}f(\sigma_2)$ *and* $\mathscr{I}^{\alpha}_{\left(\frac{\sigma_1+\sigma_2}{2}\right)^-}f(\sigma_1)$ *are called the middle starting point operators. Hence,* $\mathscr{I}^{\alpha}_{\sigma_1+}f\left(\frac{\sigma_1+\sigma_2}{2}\right)$ *and* $\mathscr{I}^{\alpha}_{\sigma_2-}f\left(\frac{\sigma_1+\sigma_2}{2}\right)$ *are called the middle ending point operators.*

The *AG*-Multiplicative Riemann-Liouville fractional integrals of order $\alpha > 0$, which are defined for a positive integrable function $f : [\sigma_1, \sigma_2] \to \mathbb{R}^+$, were introduced in [7]:

$$_{\sigma_1+}\mathscr{I}^{\alpha}_{*}f(x) = \exp\left\{\mathscr{I}^{\alpha}_{\sigma_1+}(\ln\circ f)(x)\right\},$$

$$_{*}\mathscr{I}^{\alpha}_{\sigma_2-}f(x) = \exp\left\{\mathscr{I}^{\alpha}_{\sigma_2-}(\ln\circ f)(x)\right\}.$$

Considerable research has been devoted to multiplicative integral inequalities, where Simpson, midpoint, and trapezoid-type inequalities for multiplicatively s-convex functions were established using the classical absolute value in [8], Fractional Maclaurin-type inequalities for multiplicatively convex functions and multiplicatively P-functions [9], Estimations of bounds on the multiplicative fractional integral inequalities [10] and parameterized inequalities for fractional multiplicative integrals were developed for multiplicatively s-convex and increasing functions via the classical absolute value in [11, 12, 13, 14].

Taking into account the insights from the previously mentioned works, this study commences with the introduction of three parameterized multiplicative integral identities tailored for *AG*-multiplicative differentiable functions. Expanding upon these identities, we proceed to establish a series of fractional inequalities, purposefully designed for multiplicatively h-convex functions. To conclude our investigation, we offer illustrative special cases as midpoint, trapezoid, Bullen, Milne, Simpson and corrected Simpson type inequalities in the field of *AG*-multiplicative h-convexity.

Basic parameterized identities

The following lemmas are required for the establishment of our principal results.

Lemma 15. *Let f be a positive, multiplicatively differentiable function on I° with $[a,b] \subset I^\circ$, $u : [0,1] \to [a,b]$ be a differentiable function and $g : I^\circ \to \mathbb{R}$ be a differentiable function, then $(f^* \circ u)^g$ is integrable function, and for $0 \le \xi_1 < \xi_2 \le 1$, we have*

$$\int_{\xi_1}^{\xi_2}\left((f^*(u(t)))^{u'(t)\cdot g(t)}\right)^{dt} \tag{10}$$
$$= \frac{((f\circ u)(\xi_2))^{g(\xi_2)}}{((f\circ u)(\xi_1))^{g(\xi_1)}} \cdot \frac{1}{\int_{\xi_1}^{\xi_2}\left(((f\circ u)(x))^{g'(x)}\right)^{dx}}.$$

Proof. Using the property $(f \circ u)^*(t) = (f^*(u(t)))^{u'(t)}$, we can obtain the following:

$$\int_{\xi_1}^{\xi_2}\left((f^*(u(t)))^{u'(t)\cdot g(t)}\right)^{dt} = \int_{\xi_1}^{\xi_2}\left([(f\circ u)^*(t)]^{g(t)}\right)^{dt}.$$

The application of integration by parts to multiplicative integrals results in

$$\int_{\xi_1}^{\xi_2}\left((f^*(u(t)))^{u'(t)\cdot g(t)}\right)^{dt}$$
$$= \frac{((f\circ u)(\xi_2))^{g(\xi_2)}}{((f\circ u)(\xi_1))^{g(\xi_1)}} \cdot \frac{1}{\int_{\xi_1}^{\xi_2}\left(((f\circ u)(x))^{g'(x)}\right)^{dx}}.$$

□

Lemma 16. *Let $\alpha > 0$, $\nu \in \mathbb{R}$ and $f : I^\circ \to \mathbb{R}^+$ be an AG-multiplicative differentiable mapping on $[a,b] \subset I^\circ$. If f^* is AG-multiplicatively integrable on $[a,b]$, then*

$$\frac{(f(a)f(b))^{\frac{\nu}{2}}\left(f\left(\frac{a+b}{2}\right)\right)^{1-\nu}}{\left[_{a^+}\mathscr{I}^{\alpha}_{*}f(b)\cdot {}_{*}\mathscr{I}^{\alpha}_{b^-}f(a)\right]^{\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}}}$$

$$= \left(\int_0^{\frac{1}{2}} \left[\left(F^* \left((1-t) a + t b \right) \right)^{2t^\alpha - \nu} \right]^{dt} \right)^{\frac{b-a}{4}} \tag{11}$$

$$\times \left(\int_{\frac{1}{2}}^1 \left[\left(F^* \left((1-t) a + t b \right) \right)^{2t^\alpha + \nu - 2} \right]^{dt} \right)^{\frac{b-a}{4}},$$

where

$$F(\tau) = f(\tau) \cdot f(a+b-\tau). \tag{12}$$

Proof. Employing the equality (10) yields

$$J_1 = \left(\int_0^{\frac{1}{2}} \left[\left(F^* \left((1-t) a + t b \right) \right)^{2t^\alpha - \nu} \right]^{dt} \right)^{\frac{b-a}{4}}$$

$$= \int_0^{\frac{1}{2}} \left[\left(F^* \left((1-t) a + t b \right) \right)^{(b-a)\frac{2t^\alpha - \nu}{4}} \right]^{dt}$$

$$= \frac{\left(F\left(\frac{a+b}{2}\right) \right)^{\frac{2\left(\frac{1}{2}\right)^\alpha - \nu}{4}}}{\left(F(a) \right)^{\frac{-\nu}{4}}} \times \frac{1}{\int_0^{\frac{1}{2}} \left[\left(F\left((1-t) a + t b \right) \right)^{\frac{\alpha}{2} t^{\alpha-1}} \right]^{dt}}$$

and

$$J_2 = \left(\int_{\frac{1}{2}}^1 \left[\left(F^* \left((1-t) a + t b \right) \right)^{2t^\alpha + \nu - 2} \right]^{dt} \right)^{\frac{b-a}{4}}$$

$$= \int_{\frac{1}{2}}^1 \left[\left(F^* \left((1-t) a + t b \right) \right)^{(b-a)\frac{2t^\alpha + \nu - 2}{4}} \right]^{dt}$$

$$= \frac{\left(F(b) \right)^{\frac{\nu}{4}}}{\left(F\left(\frac{a+b}{2}\right) \right)^{\frac{2\left(\frac{1}{2}\right)^\alpha + \nu - 2}{4}}} \times \frac{1}{\int_{\frac{1}{2}}^1 \left[\left(F\left((1-t) a + t b \right) \right)^{\frac{\alpha}{2} t^{\alpha-1}} \right]^{dt}}.$$

Therefore

$$J_1 \times J_2 = \frac{\left(F(b) \right)^{\frac{\nu}{4}} \left(F(a) \right)^{\frac{\nu}{4}} \left(F\left(\frac{a+b}{2}\right) \right)^{\frac{1-\nu}{2}}}{\int_0^1 \left[\left(F\left((1-t) a + t b \right) \right)^{\frac{\alpha}{2} t^{\alpha-1}} \right]^{dt}}$$

$$= \frac{\left(f(a) f(b) \right)^{\frac{\nu}{2}} \left(f\left(\frac{a+b}{2}\right) \right)^{1-\nu}}{\exp\left\{ \frac{\alpha}{2} \int_0^1 t^{\alpha-1} \ln \circ F \left((1-t) a + t b \right) dt \right\}}.$$

where

$$\ln \circ F \left((1-t) a + t b \right) = \ln \circ f \left((1-t) a + t b \right) + \ln \circ f \left(t a + (1-t) b \right).$$

Since

$$\begin{aligned} &\int_0^1 t^{\alpha-1} \ln \circ f \left((1-t) a + t b \right) dt \\ &= \frac{1}{(b-a)^\alpha} \int_a^b (s-a)^{\alpha-1} \ln \circ f(s) ds \\ &= \frac{\Gamma(\alpha)}{(b-a)^\alpha} \mathscr{I}_{b^-}^\alpha (\ln \circ f)(a) \end{aligned}$$

and

$$\begin{aligned} &\int_0^1 t^{\alpha-1} \ln \circ f \left(t a + (1-t) b \right) dt \\ &= \frac{1}{(b-a)^\alpha} \int_a^b (b-s)^{\alpha-1} \ln \circ f(s) ds \\ &= \frac{\Gamma(\alpha)}{(b-a)^\alpha} \mathscr{I}_{a^+}^\alpha (\ln \circ f)(b), \end{aligned}$$

we have

$$J_1 \times J_2 = \frac{\left(f(a) f(b) \right)^{\frac{\nu}{2}} \left(f\left(\frac{a+b}{2} \right) \right)^{1-\nu}}{\left[{}_{a^+}\mathscr{I}_*^\alpha f(b) \cdot {}_*\mathscr{I}_{b^-}^\alpha f(a) \right]^{\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}}}.$$

This gives the desired equality. $\square$

Lemma 17 (Middle starting point)**.** *Let $\alpha > 0$, $\nu \in \mathbb{R}$ and $f : I^\circ \to \mathbb{R}^+$ be an AG-multiplicative differentiable mapping on $[a,b] \subset I^\circ$. If f^* is AG-multiplicatively integrable on $[a,b]$, then*

$$\frac{\left(f(a) f(b) \right)^{\frac{\nu}{2}} \left(f\left(\frac{a+b}{2} \right) \right)^{1-\nu}}{\left[{}_{\left(\frac{a+b}{2}\right)^+}\mathscr{I}_*^\alpha f(b) \cdot {}_*\mathscr{I}_{\left(\frac{a+b}{2}\right)^-}^\alpha f(a) \right]^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}}} \tag{13}$$

$$= \left(\int_0^1 \left[\left(F^* \left(\left(1-\frac{t}{2}\right) a + \frac{t}{2} b \right) \right)^{t^{\alpha}-\nu} \right]^{dt} \right)^{\frac{b-a}{4}},$$

where

$$F(\tau) = f(\tau) \cdot f(a+b-\tau).$$

Proof. Employing the equality (10) yields

$$J_3 = \left(\int_0^1 \left[\left(F^* \left(\left(1-\frac{t}{2}\right) a + \frac{t}{2} b \right) \right)^{t^{\alpha}-\nu} \right]^{dt} \right)^{\frac{b-a}{4}}$$

$$= \int_0^1 \left[\left(F^* \left(\left(1-\frac{t}{2}\right) a + \frac{t}{2} b \right) \right)^{\frac{b-a}{2} \frac{t^{\alpha}-\nu}{2}} \right]^{dt}$$

$$= \frac{\left(F(\frac{a+b}{2})\right)^{\frac{1-\nu}{2}}}{(F(a))^{\frac{-\nu}{2}}} \times \frac{1}{\int_0^1 \left[\left(F \left(\left(1-\frac{t}{2}\right) a + \frac{t}{2} b \right) \right)^{\frac{\alpha}{2} t^{\alpha-1}} \right]^{dt}}$$

$$= \frac{(f(a)f(b))^{\frac{\nu}{2}} \left(f(\frac{a+b}{2})\right)^{1-\nu}}{\exp\left\{ \frac{\alpha}{2} \int_0^1 t^{\alpha-1} \ln\circ F \left(\left(1-\frac{t}{2}\right) a + \frac{t}{2} b \right) dt \right\}}.$$

where

$$\begin{aligned} &\ln\circ F \left(\left(1-\frac{t}{2}\right) a + \frac{t}{2} b \right) \\ &= \ln\circ f \left(\left(1-\frac{t}{2}\right) a + \frac{t}{2} b \right) \\ &+ \ln\circ f \left(\frac{t}{2} a + \left(1-\frac{t}{2}\right) b \right). \end{aligned}$$

Given that

$$\begin{aligned} &\int_0^1 t^{\alpha-1} \ln\circ f \left(\left(1-\frac{t}{2}\right) a + \frac{t}{2} b \right) dt \\ &= \left(\frac{2}{b-a}\right)^{\alpha} \int_a^{\frac{a+b}{2}} (s-a)^{\alpha-1} \ln\circ f(s) ds \\ &= \frac{2^{\alpha}\Gamma(\alpha)}{(b-a)^{\alpha}} \mathscr{I}^{\alpha}_{\left(\frac{a+b}{2}\right)^-} (\ln\circ f)(a) \end{aligned}$$

and

$$\begin{aligned} &\int_0^1 t^{\alpha-1} \ln\circ f \left(\frac{t}{2} a + \left(1-\frac{t}{2}\right) b \right) dt \\ &= \left(\frac{2}{b-a}\right)^{\alpha} \int_{\frac{a+b}{2}}^{b} (b-s)^{\alpha-1} \ln\circ f(s) ds \\ &= \frac{2^{\alpha}\Gamma(\alpha)}{(b-a)^{\alpha}} \mathscr{I}^{\alpha}_{\left(\frac{a+b}{2}\right)^+} (\ln\circ f)(b), \end{aligned}$$

we get

$$J_3 = \frac{(f(a)f(b))^{\frac{\nu}{2}} \left(f\left(\frac{a+b}{2}\right) \right)^{1-\nu}}{\left[{}_{\left(\frac{a+b}{2}\right)^+}\mathscr{I}^{\alpha}_* f(b) \cdot {}_*\mathscr{I}^{\alpha}_{\left(\frac{a+b}{2}\right)^-} f(a) \right]^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}}}.$$

□

Lemma 18 (Middle ending point)**.** *Let $\alpha > 0$, $\nu \in \mathbb{R}$ and $f : I^{\circ} \to \mathbb{R}^+$ be an AG-multiplicative differentiable mapping on $[a,b] \subset I^{\circ}$. If f^* is AG-multiplicatively integrable on $[a,b]$, then*

$$\frac{(f(a)f(b))^{\frac{\nu}{2}} \left(f\left(\frac{a+b}{2}\right) \right)^{1-\nu}}{\left[{}_{a^+}\mathscr{I}^{\alpha}_* f\left(\frac{a+b}{2}\right) \cdot {}_*\mathscr{I}^{\alpha}_{b^-} f\left(\frac{a+b}{2}\right) \right]^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}}} \tag{14}$$

$$= \left(\int_0^1 \left[\left(F^* \left(\begin{array}{c} \left(\frac{1-t}{2}\right) a \\ + \left(\frac{1+t}{2}\right) b \end{array} \right) \right)^{t^{\alpha}+\nu-1} \right]^{dt} \right)^{\frac{b-a}{4}},$$

where

$$F(\tau) = f(\tau) \cdot f(a+b-\tau).$$

Proof. Employing the equality (10) yields

$$J_4 = \left(\int_0^1 \left[\left(F^* \left(\begin{array}{c} \left(\frac{1-t}{2}\right) a \\ + \left(\frac{1+t}{2}\right) b \end{array} \right) \right)^{t^{\alpha}+\nu-1} \right]^{dt} \right)^{\frac{b-a}{4}}$$

$$= \int_0^1 \left[\left(F^* \left(\begin{array}{c} \left(\frac{1-t}{2}\right) a \\ + \left(\frac{1+t}{2}\right) b \end{array} \right) \right)^{\frac{b-a}{2} \frac{t^{\alpha}+\nu-1}{2}} \right]^{dt}$$

$$= \frac{(F(b))^{\frac{\nu}{2}}}{\left(F\left(\frac{a+b}{2}\right)\right)^{\frac{\nu-1}{2}}} \times \frac{1}{\int_0^1 \left[\left(F\left(\begin{array}{c}\left(\frac{1-t}{2}\right)a\\+\left(\frac{1+t}{2}\right)b\end{array}\right)\right)^{\frac{\alpha}{2}t^{\alpha-1}}\right]^{dt}}$$

$$= \frac{(f(a)f(b))^{\frac{\nu}{2}}\left(f\left(\frac{a+b}{2}\right)\right)^{1-\nu}}{\exp\left\{\frac{\alpha}{2}\int_0^1 t^{\alpha-1}\ln\circ F\left(\begin{array}{c}\left(\frac{1-t}{2}\right)a\\+\left(\frac{1+t}{2}\right)b\end{array}\right)dt\right\}},$$

where

$$\ln\circ F\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right) = \ln\circ f\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right) + \ln\circ f\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right).$$

Given that

$$\int_0^1 t^{\alpha-1}\ln\circ f\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)dt = \left(\frac{2}{b-a}\right)^{\alpha}\int_{\frac{a+b}{2}}^{b}\left(s-\frac{a+b}{2}\right)^{\alpha-1}\ln\circ f(s)ds = \frac{2^{\alpha}\Gamma(\alpha)}{(b-a)^{\alpha}}\mathscr{I}^{\alpha}_{b^-}(\ln\circ f)\left(\frac{a+b}{2}\right)$$

and

$$\int_0^1 t^{\alpha-1}\ln\circ f\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right)dt = \left(\frac{2}{b-a}\right)^{\alpha}\int_{a}^{\frac{a+b}{2}}\left(\frac{a+b}{2}-s\right)^{\alpha-1}\ln\circ f(s)ds = \frac{2^{\alpha}\Gamma(\alpha)}{(b-a)^{\alpha}}\mathscr{I}^{\alpha}_{a^+}(\ln\circ f)\left(\frac{a+b}{2}\right),$$

we have

$$J_4 = \frac{(f(a)f(b))^{\frac{\nu}{2}}\left(f\left(\frac{a+b}{2}\right)\right)^{1-\nu}}{\left[{}_{a^+}\mathscr{I}^{\alpha}_{*}f\left(\frac{a+b}{2}\right)\cdot *\mathscr{I}^{\alpha}_{b^-}f\left(\frac{a+b}{2}\right)\right]^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}}}.$$

□

PARAMETERIZED FRACTIONAL INEQUALITIES VIA AG-MULTIPLICATIVE h-CONVEX FUNCTIONS

Theorem 19. *Let $\alpha>0$, $\nu\in\mathbb{R}$, h be a B-function and f be a positive differentiable function on the interval on I° with $[a,b]\subset I^{\circ}$. If $|f^*|_*$ is an $*h$-convex function, then the next AG-multiplicative inequality holds.*

$$\left|\frac{(f(a)f(b))^{\frac{\nu}{2}}\left(f\left(\frac{a+b}{2}\right)\right)^{1-\nu}}{\left[{}_{a^+}\mathscr{I}^{\alpha}_{*}f(b)\cdot *\mathscr{I}^{\alpha}_{b^-}f(a)\right]^{\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}}}\right|_* \le \left[|f^*(b)|_*|f^*(a)|_*\right]^{\left(\frac{b-a}{2}\right)h\left(\frac{1}{2}\right)C_{\nu}(\alpha)}, \tag{15}$$

where

$$C_{\nu}(\alpha) = \int_0^{\frac{1}{2}}|2t^{\alpha}-\nu|dt + \int_{\frac{1}{2}}^{1}|2t^{\alpha}+\nu-2|dt. \tag{16}$$

Proof. Given that $F(\tau)=f(\tau)\cdot f(a+b-\tau)$, the application of properties I-6 and I-2 yields the following results.

$$F^*(((1-t)a+tb)) = (f^*((1-t)a+tb))(f^*(ta+(1-t)b+))^{-1}.$$

Applying the identity (11) along with the equality (1) gives

$$\frac{(f(a)f(b))^{\frac{\nu}{2}}\left(f\left(\frac{a+b}{2}\right)\right)^{1-\nu}}{\left[{}_{a^+}\mathscr{I}^{\alpha}_{*}f(b)\cdot *\mathscr{I}^{\alpha}_{b^-}f(a)\right]^{\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}}}$$

$$= \left(\int_0^{\frac{1}{2}}\left[(F^*((1-t)a+tb))^{\frac{b-a}{4}(2t^{\alpha}-\nu)}\right]^{dt}\right) \times \left(\int_{\frac{1}{2}}^{1}\left[(F^*((1-t)a+tb))^{\frac{b-a}{4}(2t^{\alpha}+\nu-2)}\right]^{dt}\right)$$

$$= \exp\left\{\int_0^{\frac{1}{2}}\frac{b-a}{4}(2t^{\alpha}-\nu)\left[\ln\circ f^*((1-t)a+tb) - \ln\circ f^*(ta+(1-t)b+)\right]dt\right\}$$

$$\times \exp\left\{\int_{\frac{1}{2}}^{1} \frac{b-a}{4}(2t^{\alpha}+\nu-2) \times [\ln\circ f^{*}((1-t)a+tb) - \ln\circ f^{*}(ta+(1-t)b+)]\,dt\right\}.$$

With the use of the multiplicative absolute value, the equality (8) and the property III-5 yield to

$$\left|\frac{(f(a)f(b))^{\frac{\nu}{2}}\left(f\left(\frac{a+b}{2}\right)\right)^{1-\nu}}{\left[{}_{a^+}\mathscr{I}_{*}^{\alpha}f(b)\cdot *\mathscr{I}_{b^-}^{\alpha}f(a)\right]^{\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}}}\right|_{*}$$

$$\leq \left|\exp\left\{\int_{0}^{\frac{1}{2}} \frac{b-a}{4}(2t^{\alpha}-\nu) \cdot [\ln\circ f^{*}((1-t)a+tb) - \ln\circ f^{*}(ta+(1-t)b+)]\,dt\right\}\right|_{*}$$

$$\times \left|\exp\left\{\int_{\frac{1}{2}}^{1} \frac{b-a}{4}(2t^{\alpha}+\nu-2) \cdot [\ln\circ f^{*}((1-t)a+tb) - \ln\circ f^{*}(ta+(1-t)b+)]\,dt\right\}\right|_{*}$$

$$\leq \exp\left\{\int_{0}^{\frac{1}{2}} \frac{b-a}{4}|2t^{\alpha}-\nu| \cdot [|\ln\circ f^{*}((1-t)a+tb)| + |\ln\circ f^{*}(ta+(1-t)b+)|]\,dt\right\}$$

$$\times \exp\left\{\int_{\frac{1}{2}}^{1} \frac{b-a}{4}|2t^{\alpha}+\nu-2| \cdot [|\ln\circ f^{*}((1-t)a+tb)| + |\ln\circ f^{*}(ta+(1-t)b+)|]\,dt\right\}$$

.

Employing (9) produces

$$\left|\frac{(f(a)f(b))^{\frac{\nu}{2}}\left(f\left(\frac{a+b}{2}\right)\right)^{1-\nu}}{\left[{}_{a^+}\mathscr{I}_{*}^{\alpha}f(b)\cdot *\mathscr{I}_{b^-}^{\alpha}f(a)\right]^{\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}}}\right|_{*}$$

$$\leq \exp\left\{\int_{0}^{\frac{1}{2}} \ln\left[|f^{*}((1-t)a+tb)|_{*} \cdot |f^{*}(ta+(1-t)b+)|_{*}\right]^{\frac{b-a}{4}|2t^{\alpha}-\nu|}dt\right\}$$

$$\times \exp\left\{\int_{\frac{1}{2}}^{1} \ln\left[|f^{*}((1-t)a+tb)|_{*} \cdot |f^{*}(ta+(1-t)b+)|_{*}\right]^{\frac{b-a}{4}|2t^{\alpha}+\nu-2|}dt\right\}.$$

Given that h satisfies the conditions of the inequality (7) and that $|f^{*}|_{*}$ has $*h$-convexity, we derive the following results.

$$\left|\frac{(f(a)f(b))^{\frac{\nu}{2}}\left(f\left(\frac{a+b}{2}\right)\right)^{1-\nu}}{\left[{}_{a^+}\mathscr{I}_{*}^{\alpha}f(b)\cdot *\mathscr{I}_{b^-}^{\alpha}f(a)\right]^{\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}}}\right|_{*}$$

$$\leq \exp\left\{\int_{0}^{\frac{1}{2}} \ln\left[|f^{*}(a)|_{*} \cdot |f^{*}(b)|_{*}\right]^{2h\left(\frac{1}{2}\right)\frac{b-a}{4}|2t^{\alpha}-\nu|}dt\right\}$$

$$\times \exp\left\{\int_{\frac{1}{2}}^{1} \ln\left[|f^{*}(a)|_{*} \cdot |f^{*}(b)|_{*}\right]^{2h\left(\frac{1}{2}\right)\frac{b-a}{4}|2t^{\alpha}+\nu-2|}dt\right\}$$

$$= \left[|f^{*}(b)|_{*}\,|f^{*}(a)|_{*}\right]^{\left(\frac{b-a}{2}\right)h\left(\frac{1}{2}\right)C_{\nu}(\alpha)}.$$

The desired inequality (15) is achieved. □

Theorem 20. *Let $\alpha>0$, $\nu\in\mathbb{R}$, h be a B-function and f be a positive differentiable function on the interval on I° with $[a,b]\subset I^{\circ}$. If $|f^{*}|_{*}$ is an $*h$-convex function, then the next AG-multiplicative inequality holds.*

$$\left|\frac{(f(a)f(b))^{\frac{\eta}{2}}\left(f\left(\frac{a+b}{2}\right)\right)^{1-\nu}}{\left[{}_{\left(\frac{a+b}{2}\right)^+}\mathscr{I}_{*}^{\alpha}f(b)\cdot *\mathscr{I}_{\left(\frac{a+b}{2}\right)^-}^{\alpha}f(a)\right]^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}}}\right|_{*} \tag{17}$$

$$\leq \left[|f^{*}(b)|_{*}\,|f^{*}(a)|_{*}\right]^{\left(\frac{b-a}{2}\right)h\left(\frac{1}{2}\right)K_{\nu}(\alpha)},$$

where

$$K_\nu(\alpha)=\int_0^1 |t^\alpha-\nu|\,dt. \tag{18}$$

Proof. Given that $F(\tau)=f(\tau)\cdot f(a+b-\tau)$, the application of properties I-6 and I-2 yields the following results.

$$\begin{aligned}&F^*\left(\left(1-\frac{t}{2}\right)a+\frac{t}{2}b\right)\\&=\left(f^*\left(\left(1-\frac{t}{2}\right)a+\frac{t}{2}b\right)\right)\\&\quad\times\left(f^*\left(\frac{t}{2}a+\left(1-\frac{t}{2}\right)b\right)\right)^{-1}.\end{aligned}$$

Applying the identity (13) along with the equality (1) gives

$$\begin{aligned}&\frac{(f(a)f(b))^{\frac{\eta}{2}}\left(f\left(\frac{a+b}{2}\right)\right)^{1-\nu}}{\left[{}_{\left(\frac{a+b}{2}\right)^+}\mathscr{I}_*^\alpha f(b)\cdot *\mathscr{I}^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a)\right]^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}}}\\&=\left(\int_0^1\left[\left(F^*\left(\left(1-\frac{t}{2}\right)a+\frac{t}{2}b\right)\right)^{t^\alpha-\nu}\right]^{dt}\right)^{\frac{b-a}{4}}\\&=\exp\left\{\int_0^1\frac{b-a}{4}(t^\alpha-\nu)\right.\\&\quad\times\left[\ln\circ f^*\left(\left(1-\frac{t}{2}\right)a+\frac{t}{2}b\right)\right.\\&\quad\left.\left.-\ln\circ f^*\left(\frac{t}{2}a+\left(1-\frac{t}{2}\right)b\right)\right]dt\right\}.\end{aligned}$$

With the use of the multiplicative absolute value, the equality (8) yield to

$$\begin{aligned}&\left|\frac{(f(a)f(b))^{\frac{\eta}{2}}\left(f\left(\frac{a+b}{2}\right)\right)^{1-\nu}}{\left[{}_{\left(\frac{a+b}{2}\right)^+}\mathscr{I}_*^\alpha f(b)\cdot *\mathscr{I}^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a)\right]^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}}}\right|_*\\&\le\left|\exp\left\{\int_0^1\frac{b-a}{4}(t^\alpha-\nu)\right.\right.\\&\quad\times\left[\ln\circ f^*\left(\left(1-\frac{t}{2}\right)a+\frac{t}{2}b\right)\right.\\&\quad\left.\left.\left.-\ln\circ f^*\left(\frac{t}{2}a+\left(1-\frac{t}{2}\right)b\right)\right]dt\right\}\right|_*\\&\le\exp\left\{\int_0^1\frac{b-a}{4}|t^\alpha-\nu|\right.\\&\quad\times\left[\left|\ln\circ f^*\left(\left(1-\frac{t}{2}\right)a+\frac{t}{2}b\right)\right|\right.\\&\quad\left.\left.+\left|\ln\circ f^*\left(\frac{t}{2}a+\left(1-\frac{t}{2}\right)b\right)\right|\right]dt\right\}.\end{aligned}$$

By applying (9) we deduce

$$\begin{aligned}&\left|\frac{(f(a)f(b))^{\frac{\eta}{2}}\left(f\left(\frac{a+b}{2}\right)\right)^{1-\nu}}{\left[{}_{\left(\frac{a+b}{2}\right)^+}\mathscr{I}_*^\alpha f(b)\cdot *\mathscr{I}^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a)\right]^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}}}\right|_*\\&\le\exp\left\{\int_0^1\ln\left[\left|f^*\left(\left(1-\frac{t}{2}\right)a+\frac{t}{2}b\right)\right|_*\right.\right.\\&\quad\left.\left.\cdot\left|f^*\left(\frac{t}{2}a+\left(1-\frac{t}{2}\right)b\right)\right|_*\right]^{\frac{b-a}{4}|t^\alpha-\nu|}dt\right\}.\end{aligned}$$

Given that $|f^*|_*$ is an $*h$-convex, we get

$$\begin{aligned}&\left|\frac{(f(a)f(b))^{\frac{\eta}{2}}\left(f\left(\frac{a+b}{2}\right)\right)^{1-\nu}}{\left[{}_{\left(\frac{a+b}{2}\right)^+}\mathscr{I}_*^\alpha f(b)\cdot *\mathscr{I}^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a)\right]^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}}}\right|_*\\&\le\exp\left\{\int_0^1\ln\left[|f^*(a)|_*\cdot|f^*(b)|_*\right]^{2h\left(\frac12\right)\frac{b-a}{4}|t^\alpha-\nu|}dt\right\}\\&=\left[|f^*(a)|_*\cdot|f^*(b)|_*\right]^{h\left(\frac12\right)\frac{b-a}{2}\int_0^1|t^\alpha-\nu|\,dt}.\end{aligned}$$

The desired inequality (17) is achieved. □

Theorem 21. *Let $\alpha>0$, $\nu\in\mathbb{R}$, h be a B-function and f be a positive differentiable function on the interval on I° with $[a,b]\subset I^\circ$. If $|f^*|_*$ is an $*h$-convex function, then the next AG-multiplicative* *inequality holds.*

$$\left| \frac{(f(a)f(b))^{\frac{\nu}{2}}\left(f(\frac{a+b}{2})\right)^{1-\nu}}{\left[{}_{a^+}\mathscr{I}_*^{\alpha} f\left(\frac{a+b}{2}\right)\cdot *\mathscr{I}_{b^-}^{\alpha} f\left(\frac{a+b}{2}\right)\right]^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}}} \right|_* \tag{19}$$

$$\leq \left[|f^*(b)|_* |f^*(a)|_*\right]^{\left(\frac{b-a}{2}\right)h\left(\frac{1}{2}\right)L_\nu(\alpha)}.$$

Where

$$L_\nu(\alpha) = \int_0^1 |t^\alpha + \nu - 1|\, dt. \tag{20}$$

Proof. The proof of Theorem 21 is similarly to the proof of Theorem 20. □

Parameterized fractional inequalities via AG-multiplicative s-convex functions

Putting $h(t) = t^s$ where $s \in (0,1]$, into inequalities (15), (17) and (19) delivers the following results.

Corollary 22. *Let $s \in (0,1]$, $\alpha > 0$, $\nu \in \mathbb{R}$ and f be a positive differentiable function on the interval on I° with $[a,b] \subset I^\circ$. If $|f^*|_*$ is an $*s$-convex function, then the next AG-multiplicative inequalities hold.*

$$\left| \frac{(f(a)f(b))^{\frac{\nu}{2}}\left(f\left(\frac{a+b}{2}\right)\right)^{1-\nu}}{\left[{}_{a^+}\mathscr{I}_*^{\alpha} f(b)\cdot *\mathscr{I}_{b^-}^{\alpha} f(a)\right]^{\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}}} \right|_*$$

$$\leq \left[|f^*(b)|_* \cdot |f^*(a)|_*\right]^{\left(\frac{b-a}{2^{s+1}}\right)C_\nu(\alpha)},$$

and

$$\left| \frac{(f(a)f(b))^{\frac{\eta}{2}}\left(f\left(\frac{a+b}{2}\right)\right)^{1-\eta}}{\left[{}_{\left(\frac{a+b}{2}\right)^+}\mathscr{I}_*^{\alpha} f(b)\cdot *\mathscr{I}_{\left(\frac{a+b}{2}\right)^-}^{\alpha} f(a)\right]^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}}} \right|_*$$

$$\leq \left[|f^*(b)|_* |f^*(a)|_*\right]^{\left(\frac{b-a}{2^{s+1}}\right)K_\nu(\alpha)}$$

and

$$\left| \frac{(f(a)f(b))^{\frac{\nu}{2}}\left(f(\frac{a+b}{2})\right)^{1-\nu}}{\left[{}_{a^+}\mathscr{I}_*^{\alpha} f\left(\frac{a+b}{2}\right)\cdot *\mathscr{I}_{b^-}^{\alpha} f\left(\frac{a+b}{2}\right)\right]^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}}} \right|_*$$

$$\leq \left[|f^*(b)|_* |f^*(a)|_*\right]^{\left(\frac{b-a}{2^{s+1}}\right)L_\nu(\alpha)}.$$

where $C_\nu(\alpha)$, $K_\nu(\alpha)$ and $L_\nu(\alpha)$ are defined by (16), (18) and (20) respectively.

Parameterized fractional inequalities via AG-multiplicative convex functions

Putting $h(t) = t$, into inequalities (15), (17) and (19) delivers the following results.

Corollary 23. *Let $\alpha > 0$, $\nu \in \mathbb{R}$ and f be a positive differentiable function on the interval on I° with $[a,b] \subset I^\circ$. If $|f^*|_*$ is a $*$-convex function, then the next AG-multiplicative inequalities hold.*

$$\left| \frac{(f(a)f(b))^{\frac{\nu}{2}}\left(f\left(\frac{a+b}{2}\right)\right)^{1-\nu}}{\left[{}_{a^+}\mathscr{I}_*^{\alpha} f(b)\cdot *\mathscr{I}_{b^-}^{\alpha} f(a)\right]^{\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}}} \right|_*$$

$$\leq \left[|f^*(b)|_* \cdot |f^*(a)|_*\right]^{\left(\frac{b-a}{4}\right)C_\nu(\alpha)},$$

and

$$\left| \frac{(f(a)f(b))^{\frac{\eta}{2}}\left(f\left(\frac{a+b}{2}\right)\right)^{1-\eta}}{\left[{}_{\left(\frac{a+b}{2}\right)^+}\mathscr{I}_*^{\alpha} f(b)\cdot *\mathscr{I}_{\left(\frac{a+b}{2}\right)^-}^{\alpha} f(a)\right]^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}}} \right|_*$$

$$\leq \left[|f^*(b)|_* |f^*(a)|_*\right]^{\left(\frac{b-a}{4}\right)K_\nu(\alpha)}$$

and

$$\left| \frac{(f(a)f(b))^{\frac{\nu}{2}}\left(f(\frac{a+b}{2})\right)^{1-\nu}}{\left[{}_{a^+}\mathscr{I}_*^{\alpha} f\left(\frac{a+b}{2}\right)\cdot *\mathscr{I}_{b^-}^{\alpha} f\left(\frac{a+b}{2}\right)\right]^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}}} \right|_*$$

$$\leq \left[|f^{*}(b)|_{*}|f^{*}(a)|_{*}\right]^{\left(\frac{b-a}{4}\right)L_{\nu}(\alpha)}.$$

where $C_{\nu}(\alpha)$, $K_{\nu}(\alpha)$ and $L_{\nu}(\alpha)$ are defined by (16), (18) and (20) respectively.

Parameterized fractional inequalities via AG-multiplicative P-functions

Setting $h(t)=1$ into inequalities (15), (17) and (19) yields the parameterized AG-multiplicative inequalities for $*P$-functions.

Corollary 24. *Let $\alpha>0$, $\nu\in\mathbb{R}$ and f be a positive differentiable function on the interval on I° with $[a,b]\subset I^{\circ}$. If $|f^{*}|_{*}$ is an $*P$-convex function, then the next AG-multiplicative inequalities hold.*

$$\left|\frac{(f(a)f(b))^{\frac{\nu}{2}}\left(f\left(\frac{a+b}{2}\right)\right)^{1-\nu}}{\left[{}_{a^{+}}\mathscr{I}_{*}^{\alpha}f(b)\cdot{}_{*}\mathscr{I}_{b^{-}}^{\alpha}f(a)\right]^{\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}}}\right|_{*}$$

$$\leq \left[|f^{*}(b)|_{*}\cdot|f^{*}(a)|_{*}\right]^{\left(\frac{b-a}{2}\right)C_{\nu}(\alpha)},$$

and

$$\left|\frac{(f(a)f(b))^{\frac{\eta}{2}}\left(f\left(\frac{a+b}{2}\right)\right)^{1-\eta}}{\left[{}_{\left(\frac{a+b}{2}\right)^{+}}\mathscr{I}_{*}^{\alpha}f(b)\cdot{}_{*}\mathscr{I}_{\left(\frac{a+b}{2}\right)^{-}}^{\alpha}f(a)\right]^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}}}\right|_{*}$$

$$\leq \left[|f^{*}(b)|_{*}|f^{*}(a)|_{*}\right]^{\left(\frac{b-a}{2}\right)K_{\nu}(\alpha)}$$

and

$$\left|\frac{(f(a)f(b))^{\frac{\nu}{2}}\left(f\left(\frac{a+b}{2}\right)\right)^{1-\nu}}{\left[{}_{a^{+}}\mathscr{I}_{*}^{\alpha}f\left(\frac{a+b}{2}\right)\cdot{}_{*}\mathscr{I}_{b^{-}}^{\alpha}f\left(\frac{a+b}{2}\right)\right]^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}}}\right|_{*}$$

$$\leq \left[|f^{*}(b)|_{*}|f^{*}(a)|_{*}\right]^{\left(\frac{b-a}{2}\right)L_{\nu}(\alpha)}.$$

where $C_{\nu}(\alpha)$, $K_{\nu}(\alpha)$ and $L_{\nu}(\alpha)$ are defined by (16), (18) and (20) respectively.

PARTICULAR CASES OF PARAMETERIZED AG-MULTIPLICATIVE FRACTIONAL INEQUALITIES

AG-multiplicative midpoint-type fractional inequalities

Taking $\nu=0$ gives

$$C(\alpha)=\int_{0}^{\frac{1}{2}}|2t^{\alpha}|\,dt+\int_{\frac{1}{2}}^{1}|2t^{\alpha}-2|\,dt,$$

$$K(\alpha)=\int_{0}^{1}|t^{\alpha}|\,dt,$$

$$L(\alpha)=\int_{0}^{1}|t^{\alpha}-1|\,dt.$$

The inequalities in Theorem 19, Theorem 20 and Theorem 21 simplify to the following AG-multiplicative midpoint-type fractional inequalities, given that $|f^{*}|_{*}$ is AG-multiplicatively h-convex function on the interval $[a,b]$.

$$\left|\frac{f\left(\frac{a+b}{2}\right)}{\left[{}_{a^{+}}\mathscr{I}_{*}^{\alpha}f(b)\cdot{}_{*}\mathscr{I}_{b^{-}}^{\alpha}f(a)\right]^{\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}}}\right|_{*} \tag{21}$$

$$\leq \left[|f^{*}(b)|_{*}|f^{*}(a)|_{*}\right]^{\left(\frac{b-a}{2}\right)h\left(\frac{1}{2}\right)C(\alpha)}.$$

And

$$\left|\frac{f\left(\frac{a+b}{2}\right)}{\left[{}_{\left(\frac{a+b}{2}\right)^{+}}\mathscr{I}_{*}^{\alpha}f(b)\cdot{}_{*}\mathscr{I}_{\left(\frac{a+b}{2}\right)^{-}}^{\alpha}f(a)\right]^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}}}\right|_{*} \tag{22}$$

$$\leq \left[|f^{*}(b)|_{*}|f^{*}(a)|_{*}\right]^{\left(\frac{b-a}{2}\right)h\left(\frac{1}{2}\right)K(\alpha)}.$$

And

$$\left|\frac{f\left(\frac{a+b}{2}\right)}{\left[{}_{a^{+}}\mathscr{I}_{*}^{\alpha}f\left(\frac{a+b}{2}\right)\cdot{}_{*}\mathscr{I}_{b^{-}}^{\alpha}f\left(\frac{a+b}{2}\right)\right]^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}}}\right|_{*} \tag{23}$$

$$\leq \left[|f^{*}(b)|_{*}|f^{*}(a)|_{*}\right]^{\left(\frac{b-a}{2}\right)h\left(\frac{1}{2}\right)L(\alpha)}.$$

Remark 25.

- *Taking $h(t)=t^s$ where $s\in(0,1]$ in then above inequalities (21), (22) and (23) gives results related to AGmultiplicative s-convex functions.*
- *Changing $h(t)=t$ in the inequalities (21), (22) and (23) yields conclusions related to AG-multiplicative convex functions.*
- *Adopting $h(t)=1$ in the inequalities (21), (22) and (23) produces consequences through AG-multiplicative P-functions convex mapping.*
- *Putting $\alpha=1$ in the preceding inequalities (21), (22) and (23) yields the midpoint inequality given in [15].*

$$\left|\frac{f\left(\frac{a+b}{2}\right)}{\left(\int_a^b (f(t))^{dt}\right)^{\frac{1}{b-a}}}\right|_* \le \left[|f^*(b)|_* \cdot |f^*(a)|_*\right]^{\frac{(b-a)h\left(\frac{1}{2}\right)}{4}}.$$

*AG-**multiplicative trapezoid-type fractional inequalities***

Taking $\nu=1$ gives

$$C(\alpha)=\int_0^1 |2t^\alpha-1|\,dt,$$
$$K(\alpha)=\int_0^1 |t^\alpha-1|\,dt,$$
$$L(\alpha)=\int_0^1 |t^\alpha|\,dt.$$

The inequalities in Theorem 19, Theorem 20 and Theorem 21 simplify to the following *AG*-multiplicative trapezoid-type fractional inequalities, given that $|f^*|_*$ is *AG*-multiplicatively *h*-convex function on the interval $[a,b]$.

$$\left|\frac{\sqrt{f(a)f(b)}}{\left[{}_{a^+}\mathscr{I}_*^\alpha f(b)\cdot{}_*\mathscr{I}_{b^-}^\alpha f(a)\right]^{\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}}}\right|_* \tag{24}$$
$$\le \left[|f^*(b)|_*\,|f^*(a)|_*\right]^{\left(\frac{b-a}{2}\right)h\left(\frac{1}{2}\right)C(\alpha)}.$$

And

$$\left|\frac{\sqrt{f(a)f(b)}}{\left[{}_{\left(\frac{a+b}{2}\right)^+}\mathscr{I}_*^\alpha f(b)\cdot{}_*\mathscr{I}_{\left(\frac{a+b}{2}\right)^-}^\alpha f(a)\right]^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}}}\right|_* \tag{25}$$
$$\le \left[|f^*(b)|_*\,|f^*(a)|_*\right]^{\left(\frac{b-a}{2}\right)h\left(\frac{1}{2}\right)K(\alpha)}.$$

And

$$\left|\frac{\sqrt{f(a)f(b)}}{\left[{}_{a^+}\mathscr{I}_*^\alpha f\left(\frac{a+b}{2}\right)\cdot{}_*\mathscr{I}_{b^-}^\alpha f\left(\frac{a+b}{2}\right)\right]^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}}}\right|_* \tag{26}$$
$$\le \left[|f^*(b)|_*\,|f^*(a)|_*\right]^{\left(\frac{b-a}{2}\right)h\left(\frac{1}{2}\right)L(\alpha)}.$$

Remark 26.

- *Taking $h(t)=t^s$ where $s\in(0,1]$ in then above inequalities (24), (25) and (26) gives results related to AGmultiplicative s-convex functions.*
- *Changing $h(t)=t$ in the inequalities (24), (25) and (26) yields conclusions related to AG-multiplicative convex functions.*
- *Adopting $h(t)=1$ in the inequalities (24), (25) and (26) produces consequences through AG-multiplicative P-functions convex mapping.*
- *Putting $\alpha=1$ in the preceding inequalities (24), (25) and (26) yields the trapezoid inequality established in [15].*

$$\left|\frac{\sqrt{f(a)f(b)}}{\left(\int_a^b (f(t))^{dt}\right)^{\frac{1}{b-a}}}\right|_*$$

$$\leq \left[|f^*(b)|_* \cdot |f^*(a)|_* \right]^{\frac{(b-a)h\left(\frac{1}{2}\right)}{4}}.$$

AG-multiplicative Bullen-type fractional inequalities

Taking $\nu = \frac{1}{2}$, we get

$$C(\alpha) = \int_0^{\frac{1}{2}} \left| 2t^\alpha - \frac{1}{2} \right| dt + \int_{\frac{1}{2}}^1 \left| 2t^\alpha - \frac{3}{2} \right| dt,$$

$$K(\alpha) = L(\alpha) = \int_0^1 \left| t^\alpha - \frac{1}{2} \right| dt.$$

The inequalities in Theorem 19, Theorem 20 and Theorem 21 simplify to the following *AG*-multiplicative Bullen-type fractional inequalities, given that $|f^*|_*$ is *AG*-multiplicatively h-convex function on the interval $[a,b]$.

$$\left| \frac{\left((f(a)f(b)) \left(f(\frac{a+b}{2}) \right)^2 \right)^{\frac{1}{4}}}{\left[{}_{a^+}\mathscr{I}_*^\alpha f(b) \cdot {}_*\mathscr{I}_{b^-}^\alpha f(a) \right]^{\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}}} \right|_* \tag{27}$$

$$\leq \left[|f^*(b)|_* |f^*(a)|_* \right]^{\left(\frac{b-a}{2}\right) h\left(\frac{1}{2}\right) C(\alpha)}.$$

And

$$\left| \frac{\left((f(a)f(b)) \left(f(\frac{a+b}{2}) \right)^2 \right)^{\frac{1}{4}}}{\left[{}_{\left(\frac{a+b}{2}\right)^+}\mathscr{I}_*^\alpha f(b) \cdot {}_*\mathscr{I}_{\left(\frac{a+b}{2}\right)^-}^\alpha f(a) \right]^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}}} \right|_* \tag{28}$$

$$\leq \left[|f^*(b)|_* |f^*(a)|_* \right]^{\left(\frac{b-a}{2}\right) h\left(\frac{1}{2}\right) K(\alpha)}.$$

And

$$\left| \frac{\left((f(a)f(b)) \left(f(\frac{a+b}{2}) \right)^2 \right)^{\frac{1}{4}}}{\left[{}_{a^+}\mathscr{I}_*^\alpha f\left(\frac{a+b}{2}\right) \cdot {}_*\mathscr{I}_{b^-}^\alpha f\left(\frac{a+b}{2}\right) \right]^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}}} \right|_* \tag{29}$$

$$\leq \left[|f^*(b)|_* |f^*(a)|_* \right]^{\left(\frac{b-a}{2}\right) h\left(\frac{1}{2}\right) L(\alpha)}.$$

Remark 27.

- *Taking $h(t) = t^s$ where $s \in (0,1]$ in then above inequalities (27), (28) and (29) gives results related to AGmultiplicative s-convex functions.*
- *Changing $h(t) = t$ in the inequalities (27), (28) and (29) yields conclusions related to AG-multiplicative convex functions.*
- *Adopting $h(t) = 1$ in the inequalities (27), (28) and (29) produces consequences through AG-multiplicative P-functions convex mapping.*
- *Putting $\alpha = 1$ in the preceding inequalities (27), (28) and (29) yields the Bullen inequality established in [15].*

$$\left| \frac{\left((f(a)f(b)) \left(f(\frac{a+b}{2}) \right)^2 \right)^{\frac{1}{4}}}{\left(\int_a^b (f(t))^{dt} \right)^{\frac{1}{b-a}}} \right|_*$$

$$\leq \left[|f^*(b)|_* \cdot |f^*(a)|_* \right]^{\frac{(b-a)h\left(\frac{1}{2}\right)}{8}}.$$

AG-multiplicative Milne-type fractional inequalities

Taking $\nu = \frac{4}{3}$, we get

$$C(\alpha) = \int_0^{\frac{1}{2}} \left| 2t^\alpha - \frac{4}{3} \right| dt + \int_{\frac{1}{2}}^1 \left| 2t^\alpha - \frac{2}{3} \right| dt,$$

$$K(\alpha) = \int_0^1 \left| t^\alpha - \frac{4}{3} \right| dt,$$

$$L(\alpha) = \int_0^1 \left| t^\alpha + \frac{1}{3} \right| dt.$$

The inequalities in Theorem 19, Theorem 20 and Theorem 21 simplify to the following *AG*-multiplicative Milne-type fractional inequalities, given that $|f^*|_*$ is *AG*-multiplicatively h-convex function on the interval $[a,b]$.

$$\left| \frac{\left((f(a)f(b))^2 \left(f(\frac{a+b}{2}) \right)^{-1} \right)^{\frac{1}{3}}}{\left[{}_{a^+}\mathscr{I}_*^\alpha f(b) \cdot {}_*\mathscr{I}_{b^-}^\alpha f(a) \right]^{\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}}} \right|_*$$

$$\leq \left[|f^*(b)|_* |f^*(a)|_* \right]^{\left(\frac{b-a}{2}\right) h\left(\frac{1}{2}\right) C(\alpha)}. \tag{30}$$

And

$$\left| \frac{\left((f(a)f(b))^2 \left(f\left(\frac{a+b}{2}\right) \right)^{-1} \right)^{\frac{1}{3}}}{\left[{}_{\left(\frac{a+b}{2}\right)^+}\mathscr{I}_*^{\alpha} f(b) \cdot * \mathscr{I}^{\alpha}_{\left(\frac{a+b}{2}\right)^-} f(a) \right]^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}}} \right|_* \tag{31}$$

$$\leq \left[|f^*(b)|_* |f^*(a)|_* \right]^{\left(\frac{b-a}{2}\right) h\left(\frac{1}{2}\right) K(\alpha)}.$$

And

$$\left| \frac{\left((f(a)f(b))^2 \left(f\left(\frac{a+b}{2}\right) \right)^{-1} \right)^{\frac{1}{3}}}{\left[{}_{a^+}\mathscr{I}_*^{\alpha} f\left(\frac{a+b}{2}\right) \cdot * \mathscr{I}^{\alpha}_{b^-} f\left(\frac{a+b}{2}\right) \right]^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}}} \right|_* \tag{32}$$

$$\leq \left[|f^*(b)|_* |f^*(a)|_* \right]^{\left(\frac{b-a}{2}\right) h\left(\frac{1}{2}\right) L(\alpha)}.$$

Remark 28.

- *Taking $h(t) = t^s$ where $s \in (0,1]$ in then above inequalities (30), (31) and (32) gives results related to AGmultiplicative s-convex functions.*

- *Changing $h(t) = t$ in the inequalities (30), (31) and (32) yields conclusions related to AG-multiplicative convex functions.*

- *Adopting $h(t) = 1$ in the inequalities (30), (31) and (32) produces consequences through AG-multiplicative P-functions convex mapping.*

- *Putting $\alpha = 1$ in the preceding inequalities (30), (31) and (32) yields Milne inequality established in [15].*

$$\left| \frac{\left((f(a)f(b))^2 \left(f\left(\frac{a+b}{2}\right) \right)^{-1} \right)^{\frac{1}{3}}}{\left(\int_a^b (f(t))^{dt} \right)^{\frac{1}{b-a}}} \right|_*$$

$$\leq \left[|f^*(b)|_* \cdot |f^*(a)|_* \right]^{\frac{5(b-a)h\left(\frac{1}{2}\right)}{12}}.$$

AG-multiplicative Simpson-type fractional inequalities

Taking $\nu = \frac{1}{3}$, we get

$$C(\alpha) = \int_0^{\frac{1}{2}} \left| 2t^{\alpha} - \frac{1}{3} \right| dt + \int_{\frac{1}{2}}^1 \left| 2t^{\alpha} - \frac{5}{3} \right| dt,$$

$$K(\alpha) = \int_0^1 \left| t^{\alpha} - \frac{1}{3} \right| dt,$$

$$L(\alpha) = \int_0^1 \left| t^{\alpha} - \frac{2}{3} \right| dt.$$

The inequalities in Theorem 19, Theorem 20 and Theorem 21 simplify to the following *AG*-multiplicative Simpson-type fractional inequalities, given that $|f^*|_*$ is *AG*-multiplicatively *h*-convex function on the interval $[a,b]$.

$$\left| \frac{\left((f(a)f(b)) \left(f\left(\frac{a+b}{2}\right) \right)^{4} \right)^{\frac{1}{6}}}{\left[{}_{a^+}\mathscr{I}_*^{\alpha} f(b) \cdot * \mathscr{I}^{\alpha}_{b^-} f(a) \right]^{\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}}} \right|_* \tag{33}$$

$$\leq \left[|f^*(b)|_* |f^*(a)|_* \right]^{\left(\frac{b-a}{2}\right) h\left(\frac{1}{2}\right) C(\alpha)}.$$

And

$$\left| \frac{\left((f(a)f(b)) \left(f\left(\frac{a+b}{2}\right) \right)^{4} \right)^{\frac{1}{6}}}{\left[{}_{\left(\frac{a+b}{2}\right)^+}\mathscr{I}_*^{\alpha} f(b) \cdot * \mathscr{I}^{\alpha}_{\left(\frac{a+b}{2}\right)^-} f(a) \right]^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}}} \right|_* \tag{34}$$

$$\leq \left[|f^*(b)|_* |f^*(a)|_* \right]^{\left(\frac{b-a}{2}\right) h\left(\frac{1}{2}\right) K(\alpha)}.$$

And

$$\left| \frac{\left((f(a)f(b)) \left(f\left(\frac{a+b}{2}\right) \right)^{4} \right)^{\frac{1}{6}}}{\left[{}_{a^+}\mathscr{I}_*^{\alpha} f\left(\frac{a+b}{2}\right) \cdot * \mathscr{I}^{\alpha}_{b^-} f\left(\frac{a+b}{2}\right) \right]^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}}} \right|_*$$

$$\leq \left[\left|f^{*}(b)\right|_{*}\left|f^{*}(a)\right|_{*}\right]^{\left(\frac{b-a}{2}\right)h\left(\frac{1}{2}\right)L(\alpha)}. \tag{35}$$

Remark 29.

- *Taking $h(t) = t^s$ where $s \in (0,1]$ in then above inequalities (33), (34) and (35) gives results related to AG-multiplicative s-convex functions.*
- *Changing $h(t) = t$ in the inequalities (33), (34) and (35) yields conclusions related to AG-multiplicative convex functions.*
- *Adopting $h(t) = 1$ in the inequalities (33), (34) and (35) produces consequences through AG-multiplicative P-functions convex mapping.*
- *Putting $\alpha = 1$ in the preceding inequalities (33), (34) and (35) yields Simpson inequality established in [15].*

$$\left|\frac{\left((f(a)f(b))\left(f\left(\frac{a+b}{2}\right)\right)^{4}\right)^{\frac{1}{6}}}{\left(\int_a^b (f(t))^{dt}\right)^{\frac{1}{b-a}}}\right|_{*}$$

$$\leq \left[\left|f^{*}(b)\right|_{*}\cdot\left|f^{*}(a)\right|_{*}\right]^{\frac{5(b-a)h\left(\frac{1}{2}\right)}{36}}.$$

AG-multiplicative corrected Simpson-type fractional inequalities

Taking $\nu = \frac{7}{15}$, we get

$$C(\alpha) = \int_0^{\frac{1}{2}} \left|2t^{\alpha} - \frac{7}{15}\right| dt + \int_{\frac{1}{2}}^{1} \left|2t^{\alpha} - \frac{23}{15}\right| dt,$$

$$K(\alpha) = \int_0^1 \left|t^{\alpha} - \frac{7}{15}\right| dt,$$

$$L(\alpha) = \int_0^1 \left|t^{\alpha} - \frac{8}{15}\right| dt.$$

The inequalities in Theorem 19, Theorem 20 and Theorem 21 simplify to the following *AG*-multiplicative Simpson-type fractional inequalities, given that $|f^*|_*$ is *AG*-multiplicatively *h*-convex function on the interval $[a,b]$.

$$\left|\frac{\left((f(a)f(b))^{7}\left(f\left(\frac{a+b}{2}\right)\right)^{16}\right)^{\frac{1}{30}}}{\left[{}_{a^+}\mathscr{I}_{*}^{\alpha} f(b)\cdot {}_{*}\mathscr{I}_{b^-}^{\alpha} f(a)\right]^{\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}}}\right|_{*} \tag{36}$$

$$\leq \left[\left|f^{*}(b)\right|_{*}\left|f^{*}(a)\right|_{*}\right]^{\left(\frac{b-a}{2}\right)h\left(\frac{1}{2}\right)C(\alpha)}.$$

And

$$\left|\frac{\left((f(a)f(b))^{7}\left(f\left(\frac{a+b}{2}\right)\right)^{16}\right)^{\frac{1}{30}}}{\left[{}_{\left(\frac{a+b}{2}\right)^+}\mathscr{I}_{*}^{\alpha} f(b)\cdot {}_{*}\mathscr{I}_{\left(\frac{a+b}{2}\right)^-}^{\alpha} f(a)\right]^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}}}\right|_{*} \tag{37}$$

$$\leq \left[\left|f^{*}(b)\right|_{*}\left|f^{*}(a)\right|_{*}\right]^{\left(\frac{b-a}{2}\right)h\left(\frac{1}{2}\right)K(\alpha)}.$$

And

$$\left|\frac{\left((f(a)f(b))^{7}\left(f\left(\frac{a+b}{2}\right)\right)^{16}\right)^{\frac{1}{30}}}{\left[{}_{a^+}\mathscr{I}_{*}^{\alpha} f\left(\frac{a+b}{2}\right)\cdot {}_{*}\mathscr{I}_{b^-}^{\alpha} f\left(\frac{a+b}{2}\right)\right]^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}}}\right|_{*} \tag{38}$$

$$\leq \left[\left|f^{*}(b)\right|_{*}\left|f^{*}(a)\right|_{*}\right]^{\left(\frac{b-a}{2}\right)h\left(\frac{1}{2}\right)L(\alpha)}.$$

Remark 30.

- *Changing $h(t) = t$ in the inequalities (36), (37) and (38) yields conclusions related to AG-multiplicative convex functions.*
- *Taking $h(t) = t^s$ where $s \in (0,1]$ in then above inequalities (36), (37) and (38) gives results related to AGmultiplicative s-convex functions.*
- *Adopting $h(t) = 1$ in the inequalities (36), (37) and (38) produces consequences through AG-multiplicative P-functions convex mapping.*

- *Putting $\alpha = 1$ in the preceding inequalities (36), (37) and (38) yields the following corrected Simpson inequality via AG-multiplicatively h-convex function.*

$$\left| \frac{\left((f(a)f(b))^{7} \left(f(\frac{a+b}{2})\right)^{16} \right)^{\frac{1}{30}}}{\left(\int_a^b (f(t))^{dt} \right)^{\frac{1}{b-a}}} \right|_{*}$$

$$\leq \left[|f^{*}(b)|_{*} \cdot |f^{*}(a)|_{*} \right]^{\frac{113(b-a)h\left(\frac{1}{2}\right)}{900}}.$$

Putting $h(t) = t$ gives the next corrected Simpson inequality via AG-multiplicatively convex function.

$$\left| \frac{\left((f(a)f(b))^{7} \left(f(\frac{a+b}{2})\right)^{16} \right)^{\frac{1}{30}}}{\left(\int_a^b (f(t))^{dt} \right)^{\frac{1}{b-a}}} \right|_{*}$$

$$\leq \left[|f^{*}(b)|_{*} \cdot |f^{*}(a)|_{*} \right]^{\frac{113(b-a)}{1800}}.$$

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