IRSTI 27.31.15 10.26577/ijmph.202617112

Entropy Solutions for p-Laplacian Type Problems with lower-order perturbation and Singular Terms in Weighted Framework

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(Received 8 January 2026; revised 20 May 2026; accepted 10 June 2026)

Abstract. This paper is devoted to the study of a class of nonlinear elliptic problems involving weighted degenerate operators of p-Laplacian type, lower-order perturbations, and singular source terms. Such problems arise in the mathematical modeling of heterogeneous media and diffusion processes characterized by nonuniform physical properties, where degeneracy and singular behavior significantly complicate the analysis. The main objective of the work is to establish the existence of entropy solutions for a broad family of weighted elliptic equations with merely integrable data. The analysis is carried out within the framework of weighted Sobolev spaces associated with Muckenhoupt weights, which provide a suitable setting for handling both degeneracy and lack of regularity. The methodology relies on the construction of appropriate approximating problems, truncation techniques, uniform a priori estimates, compactness arguments, and convergence methods adapted to entropy solutions. Under suitable assumptions on the weight function, the nonlinear perturbation term, and the singular nonlinearity, the existence of at least one entropy solution is proved. The obtained results extend several earlier existence theorems by simultaneously incorporating weighted degeneracy, lower-order perturbations, and singular reaction terms into a unified framework. This study contributes to the theory of nonlinear elliptic partial differential equations by enlarging the class of problems for which solvability can be guaranteed with low-regularity data. Moreover, the developed approach provides useful analytical tools for future investigations of weighted singular problems arising in mathematical physics, engineering, and related applied sciences.

Keywords: Entropy solutions, Existence results, Singular non-linearity, Weighted Sobolev spaces.

INTRODUCTION

Let Ω be a bounded open subset of $\mathbb{R}^N$, with $N \geq 2$, $p > 1$ and $\tau \geq 1$. This work focuses on the existence of entropy solutions to Dirichlet problems given by:

$$\begin{cases} -\mathrm{div}\left(\omega(x)\Psi(\nabla u - \Theta(u))\right) + |u|^{\tau-1}u = fh(u) & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where

$$\Psi(\xi) = |\xi|^{p-2}\xi \quad \forall \xi \in \mathbb{R}^N,$$

$\omega(x)$ is a weight function (i.e. $\omega(x)$ is a measurable almost everywhere strictly positive function on $\mathbb{R}^N$), the operator $-\mathrm{div}\left(\omega(x)\Psi(\nabla u - \Theta(u))\right)$, where Θ is a continuous function from $\mathbb{R}$ to $\mathbb{R}^N$, is a mapping defined on the weighted Sobolev spaces $W_0^{1,p}(\Omega,\omega)$ into its dual, the term $|u|^{\tau-1}u$ represents a lower-order perturbation and the right-hand side includes a non-negative datum $f \in L^1(\Omega)$ with a singular function $h(s)$ behaving like $s^{-\gamma}$ near zero, and $\gamma \geq 0$.

We begin by reviewing existing results related to problem (1), with particular emphasis on the distinction between two principal cases based on the behavior of the source term h near zero. In the non-singular case ($\gamma = 0$), when the coercivity is

degenerate with ω, the existence of entropy solutions with $f \in L^1(\Omega)$ was tackled by Sabri et al. in [1]. Similar or more general results can be found in ([2, 3, 4]).
In the singular case ($\gamma > 0$), the lack of integrability at the origin poses significant analytical challenges. Foundational contributions for such problems with L^1 data and constant weight $\omega = 1$ include [5], with more recent developments presented in [6] (see also [6, 7]). In the presence of weight (i.e. ω is not constant), the problem reduces to a model involving only the principal degenerate and possibly singular operator. This scenario has been studied in [8], where the author proved the existence of solutions to:

$$\begin{cases} -\mathrm{div}\left(\omega(x)\,|\nabla u|^{p-2}\nabla u\right) = \dfrac{f}{u^\gamma} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where f is a non-negative function belonging to a suitable Lebesgue space, by considering separately the cases $0 < \gamma < 1$, $\gamma = 1$, and $\gamma > 1$.

Our goal is to prove the existence of entropy solutions to problem (1) under suitable assumptions on the weight function and the nonlinear terms involved. The use of weighted Sobolev spaces is essential to handle the degeneracy and to ensure compactness. These spaces are well-suited to handle spatial heterogeneity and allow the application of compact embeddings and weighted Poincaré inequalities. Foundational results in this setting can be found in [9, 10, 11]. Our method combines truncation techniques, a priori estimates, and compactness arguments.

The structure of the paper is as follows. Section presents the essential preliminaries and establishes the functional setting. In Section , we formulate the main assumptions, define the notion of entropy solutions, and state the principal existence result. Section is dedicated to the proof of this result. Finally, Section offers a concrete example that demonstrates the relevance and applicability of our theoretical framework.

Preliminary results and notations

This section introduces the fundamental notions and essential properties related to weighted Sobolev spaces, particularly those involving Muckenhoupt weights. These concepts are crucial for studying elliptic partial differential equations that exhibit degeneracies or singularities.

Muckenhoupt Weights A_p

Definition 1 *Let ω be a weight function. For $1 < p < \infty$, we say that ω belongs to the Muckenhoupt class A_p (or simply $\omega \in A_p$), if there exists a constant $C > 0$ such that for every ball $B \subset \mathbb{R}^N$,*

$$\left(\frac{1}{|B|}\int_B \omega(x)\,dx\right)\left(\frac{1}{|B|}\int_B \omega(x)^{\frac{-1}{p-1}}\,dx\right)^{p-1} \le C. \tag{3}$$

Example 2 *Here are some examples of A_p weights:*

- *$\omega(x) = |x|^\alpha \in A_p$ if and only if $-N < \alpha < N(p-1)$.*
- *$\omega(x) = (1+|x|^2)^{\alpha/2} \in A_p$ for appropriate values of α.*

Weighted Lebesgue and Sobolev Spaces

Definition 3 *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and $w \in A_p$.*

- *The weighted Lebesgue space of weight ω is defined as:*

$$L^p(\Omega,\omega) := \left\{u : \Omega \to \mathbb{R}, meas., \int_\Omega |u(x)|^p \omega(x)\,dx < \infty\right\}, \tag{4}$$

where the norm is

$$\|u\|_{p,\omega} = \left(\int_\Omega |u(x)|^p \, \omega(x) dx \right)^{\frac{1}{p}}.$$

- *The weighted Sobolev space, denoted by $W^{1,p}(\Omega,\omega)$, is the space of real-valued functions $u \in L^p(\Omega,\omega)$ such that $\partial_i u \in L^p(\Omega,\omega)$ for $i = 1,\dots,N$, which is a Banach space under the norm*

$$\|u\|_{1,p,\omega} = \left(\int_\Omega |u(x)|^p \omega(x)\, dx + \int_\Omega |\nabla u|^p \omega(x)\, dx \right)^{\frac{1}{p}} \quad (5)$$

Remark 4 *Since $\omega \in L^1_{loc}(\Omega)$, we can define the subspace $X = W_0^{1,p}(\Omega,\omega)$ of $W^{1,p}(\Omega,\omega)$ as the closure of $\mathscr{C}_0^\infty(\Omega)$ with respect to the norm (5), and by the weighted Poincaré inequality (see below), its norm is equivalent to:*

$$\|u\|_X = \left(\sum_{i=1}^N \int_\Omega |\partial_i u|^p \, \omega(x) dx \right)^{\frac{1}{p}}. \quad (6)$$

Moreover, as $\omega^{\frac{-1}{p-1}} \in L^1_{loc}(\Omega)$ then X and $W^{1,p}(\Omega,\omega)$ are separable and reflexive Banach spaces.(see [12])

Weighted Poincaré Inequality And Embedding Results

Theorem 5 *([13]) Let Ω be a bounded Lipschitz domain and $\omega \in A_p$. Then there exists a constant $\mathscr{S}_p > 0$ such that for all $\varphi \in C_c^\infty(\Omega)$,*

$$\int_\Omega |\varphi(x)|^p \omega(x)\, dx \le \mathscr{S}_p^p \int_\Omega |\nabla \varphi(x)|^p \omega(x)\, dx. \quad (7)$$

Moreover, the embedding

$$W^{1,p}(\Omega,\omega) \hookrightarrow\hookrightarrow L^p(\Omega,\omega) \quad (8)$$

is compact.

Throughout this work, we assume that the weight function $\omega \in A_s$, where A_s is a subclass of A_p defined by

$$A_s := \left\{ \omega \in A_p : \omega^{-s} \in L^1(\Omega) \text{ for some } s \in \left[\frac{1}{p-1},\infty\right) \cap \left(\frac{N}{p},\infty\right) \right\}. \quad (9)$$

This subclass allows one to shift from the weighted Sobolev space into the classical Sobolev space. Indeed, we have the following embedding theorem (see [8]):

For any $\omega \in A_s$,

$$W^{1,p}(\Omega,\omega) \hookrightarrow W^{1,p_s}(\Omega) \hookrightarrow L^q(\Omega), \quad (10)$$

where $p_s = \frac{ps}{s+1} \in [1,p)$; the range of q depends on the spatial dimension N and the coefficient p_s. This embedding is continuous and, except for some limiting cases, compact. In particular, one can ensure the existence of $q > p$ such that:

$$W^{1,p}(\Omega,\omega) \hookrightarrow L^q(\Omega), \quad (11)$$

continuously. This property is crucial for establishing a priori estimates later in the analysis.

Finally, the dual space of X is equivalent to $W^{-1,p'}(\Omega,\omega^*)$ where $\omega^*(x) := \omega^{1-p'}(x)$ and $p' = \frac{p}{p-1}$ is the conjugate of p.(see [10, 11] for more details)

Technical Lemma And Notations

We recall some useful vectorial inequalities:

Lemma 6 *([14]) For each $\xi,\eta \in \mathbb{R}^N$ and $p > 1$, we find*

$$\frac{1}{p}|\xi|^p - \frac{1}{p}|\eta|^p \le |\xi|^{p-2}\xi(\xi-\eta).$$

For technical purposes, we will use the following truncation functions, for a fixed $k > 0$ and $j > 0$:

$$T_k(s) = \max(-k, \min(k,s)),$$

$$G_k(s) = s - T_k(s),$$

$$V_{k,j}(s) = \begin{cases} 1 & |s| \leq k, \\ \frac{k+j-|s|}{k} & k < |s| < 2k, \\ 0 & |s| \geq 2k, \end{cases}$$

and

$$Z_{k,j}(s) = 1 - V_{k,j}(s).$$

Moreover, we note by $\varphi^+ = \max\{\varphi, 0\}$ and $\varphi^- = -\min\{\varphi, 0\}$ the positive and the negative part of φ.

Finally, we mention that the constant C used in our paper may change from line to line and it depends only on the data of our problem but it never depends on the indices of the sequences we well often introduce.

Statement of the problem and main result

Throughout this paper, we assume that the following assumptions hold true:

(H1) Let Ω be a bounded Lipschitz open set in $\mathbb{R}^N$ $(N \geq 2)$, $1 < p < \infty$, $\tau \geq 1$ and $\omega \in A_s$.

(H2) The function $\Theta : \mathbb{R} \longrightarrow \mathbb{R}^N$ is continuous and satisfies

$$\begin{gathered} \Theta(0) = 0 \text{ and} \\ \left|\Theta(s) - \Theta(s')\right| \leq \beta |s - s'| \text{ for all } s,\ s' \in \mathbb{R}, \end{gathered} \tag{12}$$

where β is a real constant such that $0 \leq \beta < \dfrac{1}{2\mathscr{S}_p}$ and $\mathscr{S}_p$ is the Poincare constant given in Theorem 5.

(H3) The function $h \in \mathscr{C}^0([0; +\infty[; [0; +\infty])$ is a singular sourcing that is finite outside the origin such that

$$\begin{aligned} &\exists c > 0,\ \gamma \geq 0 \text{ with the property that:} \\ &h(s) \leq \frac{c}{s^\gamma}, \qquad \text{for all } s \in (0, \infty). \end{aligned} \tag{13}$$

(H4) The function f is non-negative and belongs to $L^1(\Omega)$.

We now introduce the definition of an entropy solution to (1), followed by the statement of our main result.

Definition 7 *We say that a non-negative measurable function u is an entropy solution to problem (1) if*

$$T_k(u) \in W_0^{1,p}(\Omega, \omega), \tag{14}$$

$$\omega(x)\Psi\big(\nabla T_k(u) - \Theta(T_k(u))\big) \in \prod_{i=1}^{N} L^{p'}\big(\Omega, \omega^{1-p'}\big), \tag{15}$$

$$|u|^{\tau-1} u \in L^1(\Omega), \tag{16}$$

$$f h(u) T_k(u - \varphi) \in L^1(\Omega) \tag{17}$$

and

$$\begin{aligned} &\int_\Omega \omega(x)\Psi(\nabla u - \Theta(u)) \cdot \nabla T_k(u - \varphi)\, dx \\ &\quad + \int_\Omega |u|^{\tau-1} u T_k(u - \varphi)\, dx \\ &\leq \int_\Omega f h(u) T_k(u - \varphi)\, dx \end{aligned} \tag{18}$$

for every $k > 0$ and for any $\varphi \in W_0^{1,p}(\Omega, \omega) \cap L^\infty(\Omega)$.

Our main result is the following theorem:

Theorem 8 *Assume that **(H1)**-**(H4)** hold with $0 \leq \gamma \leq 1$. Then, there exists at least one entropy solution u of problem (1).*

Proof of our result

The proof is divided into five steps.

Step 1: Approximating problems

For $n \in \mathbb{N}$, we consider the following problem

$$\begin{cases} -\mathrm{div}\,(\omega(x)\Psi(\nabla u_n - \Theta(u_n))) + T_n(|u_n|^{\tau-1} u_n) = f_n h_n(u_n) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{19}$$

where $f_n = T_n(f)$ and

$$h_n(s) = \begin{cases} T_n(h(s)) & s \geq 0, \\ \min(n, h(0)) & s < 0. \end{cases}$$

In this approximating problem, we have truncated both the lower-order term and the singularity of the right-hand side. The existence of a weak solution $u_n \in X$ to problem (19) is ensured from the classical results used in [15, 16] (see also [17, 1]). Furthermore, Well-known results (see [18]) guarantee that $u_n \in L^\infty(\Omega)$. Indeed, let $k > 0$ and using $G_k(u_n)$ as test function in (19), together with the growth of the function $t \to t\,|t|^{\tau-1}$, we find

$$\int_{\Omega(k)} \omega(x)\Psi(\nabla u_n - \Theta(u_n)) \cdot \nabla u_n \, dx \leq n^2 \int_{\Omega(k)} |G_k(u_n)| \, dx,$$

where $\Omega(k) := \{x \in \Omega : |u_n| > k\}$. We estimate the left hand-side of the former inequality by applying Lemma 6, as follows

$$\begin{aligned} &\int_{\Omega(k)} \omega(x)\Psi(\nabla u_n - \Theta(u_n)) \cdot \nabla u_n \, dx \\ &= \int_{\Omega(k)} \omega(x)\,|\nabla u_n - \Theta(u_n)|^{p-2} (\nabla u_n - \Theta(u_n)) \cdot \nabla u_n \, dx \\ &= \int_{\Omega(k)} \omega(x)\,|\nabla u_n - \Theta(u_n)|^{p-2} (\nabla u_n - \Theta(u_n)) \cdot (\nabla u_n - \Theta(u_n) + \Theta(u_n)) \, dx \\ &\geq \frac{1}{p} \int_{\Omega(k)} \omega(x)\,|\nabla u_n - \Theta(u_n)|^p \, dx - \frac{1}{p} \int_{\Omega(k)} \omega(x)\,|\Theta(u_n)|^p \, dx \\ &\geq \frac{1}{p2^{p-1}} \int_{\Omega(k)} \omega(x)\,|\nabla u_n|^p \, dx - \frac{2}{p} \int_{\Omega(k)} \omega(x)\,|\Theta(u_n)|^p \, dx \\ &\geq \left(\frac{1}{p2^{p-1}} - \frac{2\beta^p \mathscr{S}_p^p}{p} \right) \int_{\Omega(k)} \omega(x)\,|\nabla u_n|^p \, dx, \end{aligned} \tag{20}$$

where in the above inequality, we have used

$$\begin{aligned} \frac{1}{2^{p-1}}|\nabla u_n|^p &= \frac{1}{2^{p-1}}|\nabla u_n - \Theta(u_n) + \Theta(u_n)|^p \\ &\leq \frac{1}{2^{p-1}}\Big[2^{p-1}\big(|\nabla u_n - \Theta(u_n)|^p \\ &\qquad + |\Theta(u_n)|^p\big)\Big] \\ &= |\nabla u_n - \Theta(u_n)|^p + |\Theta(u_n)|^p \end{aligned}$$

and

$$\begin{aligned} &\frac{2}{p}\int_{\Omega(k)} \omega(x)|\Theta(u_n)|^p \, dx \\ &\leq \frac{2\beta^p}{p} \int_{\Omega(k)} \omega(x)|u_n|^p \, dx \\ &\leq \frac{2\beta^p \mathscr{S}_p^p}{p} \int_{\Omega(k)} \omega(x)|\nabla u_n|^p \, dx \end{aligned}$$

Consequently,

$$\left(\frac{1}{p\,2^{p-1}} - \frac{2\beta^p \mathscr{S}_p^p}{p} \right) \int_{\Omega} \omega(x)|\nabla G_k(u_n)|^p \, dx \leq n^2 \int_{\Omega(k)} |G_k(u_n)| \, dx.$$

Keep in mind that $0 \leq \beta < \dfrac{1}{2\mathscr{S}_p}$ which implies that $\dfrac{1}{2^p} > \beta^p \mathscr{S}_p^p$, then $C_p = \frac{1}{p2^{p-1}} - \frac{2\beta^p \mathscr{S}_p^p}{p} > 0$. Hence,

$$\int_{\Omega} \omega(x)\,|\nabla G_k(u_n)|^p \, dx \leq \frac{n^2}{C_p} \int_{\Omega(k)} |G_k(u_n)| \, dx. \tag{21}$$

Using Hölder's inequality with the coefficients q and q' in the right-hand side together with the continuous embedding (11), to get

$$\begin{aligned}\|G_k(u_n)\|_X^p &= \int_\Omega \omega(x)\,|\nabla G_k(u_n)|^p\,dx\\ &\le \frac{n^2}{C_p}\,|\Omega(k)|^{\frac{1}{q'}}\left(\int_\Omega |G_k(u_n)|^q\,dx\right)^{\frac{1}{q}}\\ &\le C\,|\Omega(k)|^{\frac{1}{q'}}\,\|G_k(u_n)\|_X\,,\end{aligned}\tag{22}$$

since $p>1$, then

$$\|G_k(u_n)\|_X^{p-1} \le C\,|\Omega(k)|^{\frac{1}{q'}}\,,$$

where the positive constant C is dependent on n.

For $1<k<h$, the continuous embedding (11) and the previous inequality imply that

$$\begin{aligned}(h-k)^p\,|\Omega(h)|^{\frac{p}{q}} &= \left(\int_{\Omega(h)} (h-k)^q\,dx\right)^{\frac{p}{q}}\\ &\le \left(\int_{\Omega(h)} (|u_n|-k)^q\,dx\right)^{\frac{p}{q}}\\ &\le \left(\int_{\Omega(k)} (|u_n|-k)^q\,dx\right)^{\frac{p}{q}}\\ &\le \left(\int_{\Omega(k)} |G_k(u_n)|^q\,dx\right)^{\frac{p}{q}}\\ &\le C\int_\Omega \omega(x)\,|\nabla G_k(u_n)|^p\,dx\\ &\le C\,|\Omega(k)|^{\frac{p'}{q'}}\,.\end{aligned}$$

Thus, we deduce

$$|\Omega(h)| \le \frac{C}{(h-k)^q}\,|\Omega(k)|^{\frac{qp'}{pq'}}\,.$$

Since $q>p$ implies $\frac{qp'}{pq'}>1$, we can proceed as in [18] to conclude that $u_n\in L^\infty(\Omega)$.

To demonstrate that u_n is non-negative, we take $-u_n^-$ as a test function in the weak formulation of (19), we find

$$\begin{aligned}&\int_{\{u_n\le 0\}} \omega(x)\Psi(\nabla u_n-\Theta(u_n))\cdot\nabla u_n\,dx\\ &\quad+\int_{\{u_n\le 0\}} T_n\big(|u_n|^{\tau-1}u_n\big)u_n\,dx\\ &=-\int_\Omega f_nh_n(u_n)u_n^-\,dx.\end{aligned}\tag{23}$$

Since the second term on the left-hand side is non-negative and the right-hand side is non-positive, we proceed in the same way as before to get,

$$\|u_n^-\|_X \le 0,$$

this implies that $\|u_n^-\|_X=0$, leading us to conclude that $u_n\ge 0$ almost everywhere in Ω.

Step 2: A priori estimates and basic convergence results

In this step, we will establish some a priori estimates for the solutions u_n of the problems defined in (19). These estimates will help us to prove Theorem 8.

Lemma 9 *Suppose that the hypotheses of Theorem 8 are satisfied and let u_n be a weak solution of (19). Then, for every $k>0$,*

$$\|T_k(u_n)\|_X \le C,$$

where C is a constant independent of n. Moreover, there exists a non-negative measurable function u such that

$$u_n\to u \quad \text{a.e. in } \Omega$$

and

$$T_k(u_n)\rightharpoonup T_k(u) \quad \text{weakly in } X \text{ as } n\to+\infty.$$

Proof. For $k>0$, we use $T_k(u_n)$ as a test function in the weak formulation of (19), yields

$$\begin{aligned}&\int_\Omega \omega(x)\Psi(\nabla u_n-\Theta(u_n))\cdot\nabla T_k(u_n)\,dx\\ &\quad+\int_\Omega T_n\big(|u_n|^{\tau-1}u_n\big)T_k(u_n)\,dx\\ &=\int_\Omega f_nh_n(u_n)T_k(u_n)\,dx.\end{aligned}$$

From this, (13) as well as the observation that $|u_n|^{\tau-1}u_n$ and $T_k(u_n)$ have the same sign, we derive the inequality

$$\begin{aligned}&\int_\Omega \omega(x)\Psi\big(\nabla T_k(u_n)-\Theta(T_k(u_n))\big)\cdot\nabla T_k(u_n)\,dx\\ &\le \int_\Omega f_n\big(T_k(u_n)\big)^{1-\gamma}\,dx.\end{aligned}$$

Consequently, by using the same arguments used in (20), we can conclude

$$\int_\Omega \omega(x)|\nabla T_k(u_n)|^p\,dx \le \frac{k^{1-\gamma}\|f\|_{L^1(\Omega)}}{C_p}. \quad (24)$$

Since $f \in L^1(\Omega)$, it follows that $T_k(u_n)$ is bounded in X. Thus, there exists some $v_k \in X$ such that

$$\begin{aligned} &T_k(u_n) \rightharpoonup v_k \text{ weakly in } X, \\ &T_k(u_n) \to v_k \text{ strongly in } L^p(\Omega,\omega), \text{ and a.e. in } \Omega. \end{aligned} \quad (25)$$

Next, we will demonstrate that u_n converges in measure to some measurable function u. Let $k > 0$, from (24)and the embedding $X \hookrightarrow L^1(\Omega)$, we have

$$\begin{aligned} k \text{ meas}(\{|u_n| > k\}) &= \int_{\{|u_n|>k\}} |T_k(u_n)|\,dx \\ &\le C\int_\Omega \omega(x)|\nabla T_k(u_n)|^p\,dx \\ &\le \frac{C\|f\|_{L^1(\Omega)}}{C_p}k^{1-\gamma}. \end{aligned}$$

This leads us to conclude that, for every $k > 0$,

$$\text{meas}(\{|u_n| > k\}) \le \frac{C}{k^\gamma}. \quad (26)$$

Furthermore, for any $\delta > 0$, we have

$$\begin{aligned} &\text{meas}\big(\{|u_n - u_m| > \delta\}\big) \\ &\le \text{meas}\big(\{|u_n| > k\}\big) \\ &+ \text{meas}\big(\{|u_m| > k\}\big) \\ &+ \text{meas}\big(\{|T_k(u_n) - T_k(u_m)| > \delta\}\big). \end{aligned} \quad (27)$$

Let $\varepsilon > 0$, by using (25)-(27), we can easily find some $k(\varepsilon) > 0$ such that

$$\text{meas}\big(\{|u_n - u_m| > \delta\}\big) \le \varepsilon,$$

for any $n, m \ge n_0\big(k(\varepsilon),\delta\big)$.

This shows that u_n is a Cauchy sequence in measure; therefore, it converges almost everywhere, for a subsequence, to some measurable function u. Therefore, from (25), we obtain

$$\begin{aligned} &T_k(u_n) \rightharpoonup T_k(u) \text{ weakly in } X, \\ &T_k(u_n) \to T_k(u) \text{ strongly in } L^p(\Omega,\omega) \text{ and a.e. in } \Omega. \end{aligned} \quad (28)$$

□

On top of that, using the condition (12) and weighted Poincaré's inequality, we have

$$\begin{aligned} &\int_\Omega \omega^{1-p'}(x)\,|\omega(x)\Psi(\nabla T_k(u_n) - \Theta(T_k(u_n))|^{p'}\,dx \\ &= \int_\Omega \omega(x)\,|\nabla T_k(u_n) - \Theta(T_k(u_n))|^p\,dx \\ &\le C\int_\Omega \omega(x)\,|\nabla T_k(u_n)|^p\,dx. \end{aligned}$$

Thus, from Lemma 9, the sequence $(\omega(x)\Psi(\nabla T_k(u_n) - \Theta(T_k(u_n)))_n$ is bounded in $\prod_{i=1}^N L^{p'}(\Omega,\omega^{1-p'})$. Therefore, for every $k > 0$, there exists a function $\psi_k \in \prod_{i=1}^N L^{p'}(\Omega,\omega^{1-p'})$ such that

$$\begin{aligned} &\omega(x)\Psi\big(\nabla T_k(u_n) - \Theta(T_k(u_n))\big) \rightharpoonup \psi_k \\ &\text{weakly in } \prod_{i=1}^N L^{p'}(\Omega,\omega^{1-p'}). \end{aligned} \quad (29)$$

Step 3: Strong convergence of $\Big(T_n(|u_n|^{\tau-1}u_n)\Big)_n$

Lemma 10 *Let u_n be a weak solution of (19). Then,*

$$\begin{aligned} &T_n\big(|u_n|^{\tau-1}u_n\big) \longrightarrow |u|^{\tau-1}u \\ &\text{strongly in } L^1(\Omega). \end{aligned} \quad (30)$$

Proof. For $\delta, j > 0$, using $Z_{\delta,j}(u_n)$ as a test function in the weak formulation of (19) and ignoring the first term which is positive (recall that $Z'_{\delta,j}(u_n) \ge 0$ when $u_n \ge 0$). This yields

$$\int_\Omega T_n\big(|u_n|^{\tau-1}u_n\big)Z_{\delta,j}(u_n)\,dx \le \left(\sup_{s\in[\delta,\infty)} h(s)\right)\int_{\{u_n>\delta\}} f\,dx$$

letting $j\to 0$, we obtain

$$\int_{\{u_n>\delta\}}\left|T_n(|u_n|^{\tau-1}u_n)\right|dx \le \left(\sup_{s\in[\delta,\infty)} h(s)\right)\int_{\{u_n>\delta\}} f\,dx.$$

The fact that $f\in L^1(\Omega)$ allows us to say that, for any given $\varepsilon>0$, there exists a δ_ε such that

$$\int_{\{u_n>\delta_\varepsilon\}} f\,dx \le \varepsilon. \tag{31}$$

By (31), for any measurable subset E of Ω, It follows that

$$\int_E\left|T_n(|u_n|^{\tau-1}u_n)\right|dx = \int_{E\cap\{u_n>\delta_\varepsilon\}}\left|T_n(|u_n|^{\tau-1}u_n)\right|dx + \int_{E\cap\{u_n\le\delta_\varepsilon\}}\left|T_n(|u_n|^{\tau-1}u_n)\right|dx \le \varepsilon, \tag{32}$$

$$\int_E\left|T_n(|u_n|^{\tau-1}u_n)\right|dx = \int_{E\cap\{u_n>\delta_\varepsilon\}}\left|T_n(|u_n|^{\tau-1}u_n)\right|dx + \int_{E\cap\{u_n\le\delta_\varepsilon\}}\left|T_n(|u_n|^{\tau-1}u_n)\right|dx \le \varepsilon, \tag{33}$$

provided that meas(E) is small enough. Therefore, $T_n(|u_n|^{\tau-1}u_n))$ is equi-integrable, hence, by virtue of Lemma 9 and Vitali's theorem, it converges strongly to $|u|^{\tau-1}u$ in $L^1(\Omega)$. □

Step 4: Strong convergence of truncations

Before demonstrating the strong convergence of truncations, we need to establish the following lemma, which will be helpful in subsequent arguments.

Lemma 11 *For $k>0$, suppose that the hypotheses of Theorem 8 are satisfied and let u_n be a weak solution of (19). Then, for all non-negative $\varphi\in W_0^{1,p}(\Omega,\omega)\cap L^\infty(\Omega)$, we have*

$$\int_\Omega f_n h_n(u_n)\varphi\,dx \le C, \tag{34}$$

where the constant C does not depend on n.

Proof. Let $k>0$ and φ be a non-negative function in $W_0^{1,p}(\Omega,\omega)\cap L^\infty(\Omega)$. Taking $V_k(u_n)\varphi$ as a test function in (19) (where $V_k(u_n)=V_{k,k}(u_n)$) and ignoring the positive terms, we obtain

$$\int_\Omega f_n h_n(u_n)\chi_{\{u_n\le k\}}\varphi\,dx \le \int_\Omega \omega(x)V_k(u_n)\Psi(\nabla u_n-\Theta(u_n))\cdot\nabla\varphi\,dx + \int_\Omega T_n(|u_n|^{\tau-1}u_n)V_k(u_n)\varphi\,dx. \tag{35}$$

On the one hand, applying Young's inequality and using (12), we derive

$$\begin{aligned}\int_\Omega \omega(x)V_k(u_n)\Psi(\nabla u_n-\Theta(u_n))\cdot\nabla\varphi\,dx &\le \int_\Omega \omega(x)\Psi(\nabla T_{2k}(u_n)-\Theta(T_{2k}(u_n)))\cdot\nabla\varphi\,dx\\ \le \frac{1}{p'}\int_\Omega \omega(x)|\nabla T_{2k}(u_n)-\Theta(T_{2k}(u_n))|^p\,dx &+ \frac1p\int_\Omega \omega(x)|\nabla\varphi|^p\,dx \le C\int_\Omega \omega(x)|\nabla T_{2k}(u_n)|^p\,dx\\ +C\int_\Omega \omega(x)|T_{2k}(u_n)|^p\,dx &+ \frac1p\int_\Omega \omega(x)|\nabla\varphi|^p\,dx \le C\left(\|T_{2k}(u_n)\|_X+\|\varphi\|_X\right).\end{aligned} \tag{36}$$

On the other hand, one has

$$\int_\Omega T_n(|u_n|^{\tau-1}u_n)V_k(u_n)\varphi\,dx = \int_{u_n\le 2k} T_n(|u_n|^{\tau-1}u_n)V_k(u_n)\varphi\,dx \le C. \tag{37}$$

Combining (35)-(37), and applying Lemma 9, we obtain

$$\int_\Omega f_n h_n(u_n)\chi_{\{u_n\le k\}}\varphi\,dx \le C. \tag{38}$$

Moreover, it is straightforward to show, due to the assumptions on h and f, that

$$\int_\Omega f_n h_n(u_n)\chi_{\{u_n>k\}}\varphi\,dx \le C. \tag{39}$$

Thus, from (38) and (39), we conclude that $f_n h_n(u_n)\varphi \in L^1(\Omega)$. □

Now, we have all the ingredients to demonstrate the strong convergence of the truncations.

Lemma 12 *For $k>0$, suppose that the hypotheses of Theorem 8 are satisfied and let u_n be a weak solution of (19). Then $T_k(u_n)\to T_k(u)$ strongly in X as $n\to\infty$, where u is given by Lemma 9.*

Proof. Let $k>0$, and without loss of generality, we may assume that $n>k$. For simplicity, we set throughout the proof

$$\Phi(s,\xi)=\omega(x)\Psi(\xi-\Theta(s)).$$

We choose $T_k(u_n)-T_k(u)$ as the test function in (19), this gives

$$\begin{aligned}&\int_\Omega \Phi(u_n,\nabla u_n)\cdot\nabla(T_k(u_n)-T_k(u))\,dx\\&+\int_\Omega T_n(|u_n|^{\tau-1}u_n)(T_k(u_n)-T_k(u))\,dx\\&=\int_\Omega f_n h_n(u_n)(T_k(u_n)-T_k(u))\,dx.\end{aligned}$$

From this, we derive the equality

$$\begin{aligned}&\int_\Omega(\Phi(T_k(u_n),\nabla T_k(u_n))-\Phi(T_k(u_n),\nabla T_k(u)))\cdot\nabla(T_k(u_n)-T_k(u))\,dx\\&=-\int_\Omega\Phi(T_k(u_n),\nabla T_k(u))\cdot\nabla(T_k(u_n)-T_k(u))\,dx+\int_\Omega f_n h_n(u_n)(T_k(u_n)-T_k(u))\,dx\\&-\int_\Omega T_n(|u_n|^{\tau-1}u_n)(T_k(u_n)-T_k(u))\,dx=:\mathscr{I}_1+\mathscr{I}_2+\mathscr{I}_3.\end{aligned}\tag{40}$$

We proceed to analyze each term $(\mathscr{I}_i)_{i=1,2,3}$ in equation (40) individually to determine their limits as $j\to\infty$ and $n\to\infty$.

For the first term $\mathscr{I}_1$, by virtue of Vitali's theorem, we obtain

$$\Phi(T_k(u_n),\nabla T_k(u))\longrightarrow\Phi(T_k(u),\nabla T_k(u))\quad\text{strongly in }\prod_{i=1}^N L^{p'}(\Omega,\omega^{1-p'}).$$

Given that $T_k(u_n)\rightharpoonup T_k(u)$ weakly in X, it follows that

$$\lim_{n\to\infty}\mathscr{I}_1=0. \tag{41}$$

Turning to $\mathscr{I}_2$, we let δ be a small positive number such that

$$\delta\notin\{\eta>0:\operatorname{meas}(\{u=\eta\})>0\}.$$

Splitting $\mathscr{I}_2$ over $\{u_n\le\delta\}$ and $\{u_n>\delta\}$ and utilizing condition (13), we have

$$\begin{aligned}\mathscr{I}_2\le{}&C\delta^{1-\gamma}\int_{\{u_n\le\delta\}}f\,dx\\&+\int_{\{u_n>\delta\}}f_n h_n(u_n)\big(T_k(u_n)-T_k(u)\big)\,dx.\end{aligned}\tag{42}$$

For the first term on the right-hand side of (42), if $h(0)<\infty$ (i.e., $\gamma=0$), then $\delta^{1-\gamma}\int_{\{u_n\le\delta\}}f$ converges to zero as $n\to\infty$ and $\delta\to0^+$. If $h(0)=\infty$ (i.e., $0<\gamma\le1$), then by Fatou's lemma and (34), we see that $fh(u)\in L^1_{loc}(\Omega)$, which implies $\{u=0\}\subset\{f=0\}$ up to a set of zero Lebesgue measure. Thus, in both cases,

$$\begin{aligned}\limsup_{\delta\to0^+}\limsup_{n\to\infty}\left(\delta^{1-\gamma}\int_{\{u_n\le\delta\}}f\right)={}&\\ \limsup_{\delta\to0^+}\left(\delta^{1-\gamma}\int_{\{u\le\delta\}}f\right)={}&0.\end{aligned}\tag{43}$$

For the second term on the left-hand side of (42), using the Dominated Convergence Theorem, we have

$$\lim_{n\to\infty}\int_{\{u_n>\delta\}} f_n h_n(u_n)\left(T_k(u_n)-T_k(u)\right)dx=0. \tag{44}$$

Combining (42), (43) and (44), we conclude that

$$\limsup_{\delta\to 0^+}\limsup_{n\to\infty}\mathscr{I}_2\le 0. \tag{45}$$

Finally, concerning the term $\mathscr{I}_3$, invoking the strong convergence (30) together with $T_k(u_n)\rightharpoonup T_k(u)$ weakly in X, we deduce that

$$\lim_{n\to\infty}\mathscr{I}_3=0. \tag{46}$$

Bringing together (41), (45) and (46) in (40), we deduce

$$\lim_{n\to\infty}\int_\Omega\left(\Phi\left(T_k(u_n),\nabla T_k(u_n)\right)-\Phi\left(T_k(u_n),\nabla T_k(u)\right)\right)\cdot\nabla\left(T_k(u_n)-T_k(u)\right)dx=0. \tag{47}$$

Thus, by Lemma 3.2 in [2] and the following strict monotonicity condition

$$\Big(\Psi(\xi-\Theta(s))-\Psi(\xi'-\Theta(s))\Big)\cdot(\xi-\xi')>0,\qquad\forall\xi\neq\xi'.$$

we conclude that

$$T_k(u_n)\to T_k(u)\text{ strongly in }X\text{ as }n\to\infty. \tag{48}$$

In particular, there exists a subsequence still labelled with n such that

$$\nabla u_n\to\nabla u\text{ a.e. in }\Omega. \tag{49}$$

□

Step 5: Proof of Theorem 8

First, in Lemma 9, we established that the limit u of the approximate sequence u_n from (19) satisfies, for all $k>0$, $T_k(u)\in W_0^{1,p}(\Omega,\omega)$, verifying assertion (14).

Next, using Lemma 9, (48), (29) and noting that Ψ and Θ are continuous functions, we conclude that

$$\omega(x)\Psi(\nabla T_k(u)-\Theta(T_k(u)))\in\prod_{i=1}^{N}L^{p'}(\Omega,\omega^{1-p'}),$$

hence verifying assertion (15). Furthermore, it follows from Lemma 10 that (16) holds.

We now proceed to show assertion (17). Let $\varphi\in W_0^{1,p}(\Omega,\omega)\cap L^\infty(\Omega)$ be fixed and set $M=\|\varphi\|_{L^\infty(\Omega)}+k$. Taking $T_k(u_n-\varphi)^+$ as a test function in (19), we obtain

$$\begin{aligned}&\int_\Omega f_n h_n(u_n)T_k(u_n-\varphi)^+\,dx\\&=\int_\Omega\omega(x)\Psi(\nabla u_n-\Theta(u_n))\cdot\nabla T_k(u_n-\varphi)^+\,dx\\&\quad+\int_\Omega T_n(|u_n|^{\tau-1}u_n)T_k(u_n-\varphi)^+\,dx\\&\le\int_{\{\varphi<u_n<\varphi+k\}}\Big|\omega(x)\Psi\big(\nabla T_M(u_n)\\&\quad-\Theta(T_M(u_n))\big)\cdot\nabla\big(T_M(u_n)-\varphi\big)\Big|\,dx+C,\end{aligned}$$

which implies that

$$\begin{aligned}&\int_\Omega f_n h_n(u_n)T_k(u_n-\varphi)^+\,dx\\&\le\int_\Omega\Big|\omega(x)\Psi(\nabla T_M(u_n)\\&\quad-\Theta(T_M(u_n)))\cdot\nabla\varphi\Big|\,dx\\&\quad+\int_\Omega\Big|\omega(x)\Psi(\nabla T_M(u_n)-\Theta(T_M(u_n)))\\&\quad\cdot\nabla T_M(u_n)\Big|\,dx+C.\end{aligned}\tag{50}$$

For the first term on the right-hand side of (50), using Young's inequality and (12), we find

$$\begin{aligned}&\int_\Omega|\omega(x)\Psi(\nabla T_M(u_n)-\Theta(T_M(u_n)))\cdot\nabla\varphi|\,dx\\&\le\frac{1}{p'}\int_\Omega\omega(x)\,|\nabla T_M(u_n)-\Theta(T_M(u_n))|^p\,dx\\&+\frac{1}{p}\int_\Omega\omega(x)\,|\nabla\varphi|^p\,dx\le C\left(\|T_M(u_n)\|_X+\|\varphi\|_X\right)\le C,\end{aligned}\tag{51}$$

where C is a constant independent of n by Lemma 9.

Similarly, for the second term in (50), we find

$$\int_\Omega \Big|\omega(x)\Psi\big(\nabla T_M(u_n)-\Theta(T_M(u_n))\big) \cdot\nabla T_M(u_n)\Big|\,dx \le C. \tag{52}$$

Combining (50), (51) and (52), we obtain

$$\int_\Omega f_n h_n(u_n) T_k(u_n-\varphi)^+\,dx \le C.$$

Applying Fatou's lemma to the above inequality yields that $fh(u)T_k(u-\varphi)^+ \in L^1(\Omega)$. A similar argument with $T_k(u_n-\varphi)^-$ shows that $fh(u)T_k(u-\varphi)^- \in L^1(\Omega)$, so $fh(u)T_k(u-\varphi) \in L^1(\Omega)$.

To establish assertion (18), we take $T_k(u-\varphi)$ as a test function in (19), where $\varphi \in W_0^{1,p}(\Omega,\omega)\cap L^\infty(\Omega)$, giving us

$$\int_\Omega \omega(x)\Psi(\nabla u_n-\Theta(u_n))\cdot\nabla T_k(u_n-\varphi)\,dx + \int_\Omega T_n(|u_n|^{\tau-1}u_n)T_k(u_n-\varphi)\,dx = \int_\Omega f_n h_n(u_n)T_k(u_n-\varphi)\,dx. \tag{53}$$

Our objective is to take the limit as $n\to\infty$ in this expression. For the first term on the left-hand side, note that $\nabla T_k(u_n-\varphi)$ is nonzero only on the set $\{|u_n-\varphi|<k\}$. Setting $M=\|\varphi\|_{L^\infty(\Omega)}+k$, Lemma 12 shows that

$$T_k(u_n)\to T_k(u) \quad \text{strongly in } X \text{ as } n\to\infty.$$

Since, as $n\to\infty$,

$$\omega(x)\Psi\big(\nabla T_M(u_n)-\Theta(T_M(u_n))\big) \rightharpoonup \omega(x)\Psi\big(\nabla T_M(u)-\Theta(T_M(u))\big) \quad \text{weakly in } \prod_{i=1}^N L^{p'}(\Omega,\omega^{1-p'}). \tag{54}$$

Thus,

$$\lim_{n\to\infty}\int_\Omega \omega(x)\Psi(\nabla u_n-\Theta(u_n))\cdot\nabla T_k(u_n-\varphi)\,dx = \int_\Omega \omega(x)\Psi(\nabla u-\Theta(u))\cdot\nabla T_k(u-\varphi)\,dx. \tag{55}$$

For the second term on the left-hand side, we use the strong convergence $T_n(|u_n|^{\tau-1}u_n)$ to $|u|^{\tau-1}u$ in $L^1(\Omega)$ along with the weak$*$ convergence of $T_k(u_n-\varphi)$ to $T_k(u-\varphi)$ in $L^\infty(\Omega)$, to get

$$\lim_{n\to\infty}\int_\Omega T_n(|u_n|^{\tau-1}u_n)T_k(u_n-\varphi)\,dx = \int_\Omega |u|^{\tau-1}u\,T_k(u-\varphi)\,dx. \tag{56}$$

For the right-hand side, if $h(0)<\infty$, the limit follows by the Dominated Convergence Theorem. If $h(0)=\infty$, we decompose the right-hand side of (53) as

$$\int_\Omega f_n h_n(u_n)T_k(u_n-\varphi)\,dx = \int_\Omega f_n h_n(u_n)T_k(u_n-\varphi)^+\,dx + \int_\Omega f_n h_n(u_n)T_k(u_n-\varphi)^-\,dx.$$

Now, we pass to the limit in the first term on the right-hand side of the previous inequality, and we can treat the second term in a similar manner.

Let ε be small enough such that $\varepsilon\notin\{\eta>0:\text{meas}(\{u=\eta\})>0\}$, which is at most a countable set. By taking $(u_n-\varphi)^+V_\varepsilon(u_n)$ as a test function in (19) and disregarding the positive and negative terms, we obtain

$$\int_{\{u_n\le\varepsilon\}} f_n h_n(u_n)(u_n-\varphi)^+\,dx \le \int_\Omega V_\varepsilon(u_n)\omega(x)\Psi(\nabla u_n-\Theta(u_n))\cdot\nabla(u_n-\varphi)^+\,dx + \int_\Omega V_\varepsilon(u_n)T_n\left(|u_n|^{q-1}u_n\right)(u_n-\varphi)^+\,dx. \tag{57}$$

Using (12) and Lemma 9, it is straightforward to show that $V_\varepsilon(u_n)\omega(x)\Psi(\nabla u_n-\Theta(u_n))$ is bounded in $\prod_{i=1}^N L^{p'}(\Omega,\omega^{1-p'})$. Hence, $V_\varepsilon(u_n)\omega(x)\Psi(\nabla u_n-\Theta(u_n))$ converges weakly to $V_\varepsilon(u)\omega(x)\Psi(\nabla u-\Theta(u))$ in $\prod_{i=1}^N L^{p'}(\Omega,\omega^{1-p'})$.

Moreover, $T_{2\varepsilon}(u_n)$ converges strongly to $T_{2\varepsilon}(u)$

in X as $n \to \infty$. Therefore,

$$\lim_{n\to\infty} \int_{\{u_n \le \varepsilon\}} f_n h_n(u_n)(u_n - \varphi)^+ \, dx \le \int_\Omega V_\varepsilon(u)\omega(x)\Psi(\nabla u - \Theta(u)) \cdot \nabla(u-\varphi)^+ \, dx + \int_\Omega V_\varepsilon(u) \, |u|^{q-1} \, u(u-\varphi)^+ \, dx := C_{1,\varepsilon} + C_{2,\varepsilon}, \tag{58}$$

where

$$\lim_{\varepsilon\to 0^+} C_{1,\varepsilon} = \int_{\{u=0\}} \omega(x)\Psi(\nabla u - \Theta(u)) \cdot \nabla(u-\varphi)^+ \, dx = 0 \tag{59}$$

and

$$\lim_{\varepsilon\to 0^+} C_{2,\varepsilon} = \int_{\{u=0\}} |u|^{q-1} \, u(u-\varphi)^+ \, dx = 0, \tag{60}$$

since $\Theta(0) = 0$ and $\Psi(0) = 0$.

We decompose further as follows

$$\int_\Omega f_n h_n(u_n) T_k(u_n - \varphi)^+ \, dx = \int_{\{u_n \le \varepsilon\}} f_n h_n(u_n)(u_n - \varphi)^+ \, dx + \int_{\{u_n > \varepsilon\}} f_n h_n(u_n) T_k(u_n - \varphi)^+ \, dx. \tag{61}$$

for $k > \varepsilon$. Regarding the second term on the right-hand side of the previous inequality, we find that

$$f_n h_n(u_n) T_k(u_n - \varphi)^+ \le k \sup_{s\in(\varepsilon,\infty)} h(s) \, f,$$

invoking the Dominated Convergence Theorem, we deduce

$$\lim_{n\to\infty} \int_{\{u_n > \varepsilon\}} f_n h_n(u_n) T_k(u_n - \varphi)^+ \, dx = \int_{\{u > \varepsilon\}} f \, h(u) \, T_k(u-\varphi)^+ \, dx.$$

Furthermore, we have already shown that $f h(u) T_k(u-\varphi)^+ \in L^1(\Omega)$. Thus, a second application of the Dominated Convergence Theorem yields

$$\lim_{\varepsilon\to 0^+} \lim_{n\to\infty} \int_{\{u_n > \varepsilon\}} f_n h_n(u_n) T_k(u_n - \varphi)^+ \, dx = \int_\Omega f \, h(u) \, T_k(u-\varphi)^+ \, dx. \tag{62}$$

since it follows from $f h(u) T_k(u-\varphi) \in L^1(\Omega)$ that $\{u = 0\} \subset \{f = 0\}$.

Taking the limits as $n \to \infty$ and then $\varepsilon \to 0^+$ in (61), by using (58), (59), (60) and (62), we obtain

$$\lim_{n\to\infty} \int_\Omega f_n h_n(u_n) T_k(u_n - \varphi)^+ \, dx = \int_\Omega f \, h(u) \, T_k(u-\varphi)^+ \, dx.$$

Finally, by reasoning in the same manner as before, we conclude that

$$\lim_{n\to\infty} \int_\Omega f_n h_n(u_n) T_k(u_n - \varphi)^- \, dx = \int_\Omega f \, h(u) \, T_k(u-\varphi)^- \, dx.$$

which is sufficient to pass to the limit on the right-hand side of (53), thus proving assertion (18).

So, with this last step the proof of Theorem 8 is concluded.

Example

In this final section, we provide an explicit example to illustrate the applicability of our main result. To do so, we will take $\Omega = B_{\mathbb{R}^3}(0,1)$, where $B_{\mathbb{R}^3}(0,1)$ is the unit ball of $\mathbb{R}^3$, $p = 3$, $\tau = 2$, the weight function $\omega(x) = |x|$ which clearly satisfies $\omega \in A_3$. Consider the Lipschitz continuous function:

$$\Theta(s) = \beta s, \text{ where } \beta = \left(\frac{1}{4\mathscr{S}_3}, \frac{1}{4\mathscr{S}_3}, \frac{1}{4\mathscr{S}_3} \right).$$

Finally, if we take, for example,

$$f(x) = \exp^{-|x|^2} \text{ and } h(s) = \frac{1}{\sqrt{s}} \text{ for } s > 0,$$

then, all hypotheses of Theorem 8 are satisfied.

Therefore, problem

$$\begin{cases} -\operatorname{div}\left(|x|\left|\begin{pmatrix} \frac{\partial u}{\partial x_1}-\frac{u}{4\mathscr{S}_3} \\ \frac{\partial u}{\partial x_2}-\frac{u}{4\mathscr{S}_3} \\ \frac{\partial u}{\partial x_3}-\frac{u}{4\mathscr{S}_3} \end{pmatrix}\right|\begin{pmatrix} \frac{\partial u}{\partial x_1}-\frac{u}{4\mathscr{S}_3} \\ \frac{\partial u}{\partial x_2}-\frac{u}{4\mathscr{S}_3} \\ \frac{\partial u}{\partial x_3}-\frac{u}{4\mathscr{S}_3} \end{pmatrix}\right)+ \\ \qquad +u^2=\frac{\exp(-|x|^2)}{\sqrt{u}}, \qquad \text{in } \Omega, \\ u\geq 0, \qquad \text{in } \Omega, \\ u=0 \qquad \text{on } \partial\Omega. \end{cases} \tag{63}$$

has at least one solution.

Conclusions

In this paper, we studied a class of nonlinear weighted elliptic problems involving a p-Laplacian-type operator, a lower-order perturbation term, and a singular nonlinearity. Using approximation procedures, truncation techniques, a priori estimates, and compactness arguments in weighted Sobolev spaces, we established the existence of entropy solutions for non-negative L^1-data when $(0\leq\gamma\leq 1)$. The obtained results validate the proposed framework and extend several known existence results by simultaneously treating weighted degeneracy, singular source terms, and lower-order perturbations. This contribution broadens the theory of nonlinear elliptic equations with low-regularity data. Future work may focus on uniqueness, regularity properties, and extensions to more general weighted operators and stronger singularities.

Author Contributions: All authors contributed equally to this work. All authors participated in the conceptualization of the study, methodology development, investigation, formal analysis, data curation, validation, visualization, and manuscript preparation. All authors were involved in writing, reviewing, and editing the manuscript and approved the final version for publication. All authors have read and agreed to the published version of the manuscript.

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