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Numerical solution of the inverse problem of magnetotelluric sounding

Abstract. This study focuses on the coefficient inverse problem arising in magnetotelluric (MT) sounding, which plays a crucial role in geophysical exploration and subsurface characterization. The main objective is twofold: first, to construct a reliable forward numerical model based on the Helmholtz equation with a complex-valued conductivity coefficient, and second, to develop a stable inversion procedure for reconstructing the conductivity distribution from boundary measurements. The forward problem is discretized using a finite-difference approximation, ensuring numerical stability and accuracy for both the direct and adjoint formulations. To address the ill-posed nature of the inverse problem, a misfit functional is introduced, measuring the discrepancy between simulated and observed boundary data. This functional is minimized using the iterative Landweber method, which provides a simple yet robust tool for stabilizing reconstructions. Numerical experiments are carried out for a synthetic conductivity model consisting of a smooth background medium with an embedded localized anomaly. The obtained results demonstrate the ability of the proposed method to recover key structural features of the anomaly. The presented framework offers a promising foundation for the development of practical inversion algorithms applicable to real geophysical MT data.
Keywords: Helmholtz equation, magnetotelluric sounding, inverse problem, Landweber method, numerical solution.

Introduction

A key direction in the study of magnetotelluric processes is the construction of mathematical models that reliably describe the propagation of the electromagnetic field in a conducting medium. In [1], a generalized methodological framework is presented, where the formulation of forward and inverse problems for Maxwell's equations is discussed, and the specific features of their solutions under various physical and geometrical parameters are analyzed. Significant attention is devoted to the classification of models – from one-dimensional to multilayered and anisotropic structures – which plays an essential role in the selection of appropriate numerical approaches. Furthermore, that study highlights the relationship between the full system of Maxwell's equations and simplified scalar formulations, such as the Helmholtz equation, thereby opening the possibility of employing advanced numerical methods commonly used for elliptic problems in the context of magnetotelluric sounding.

The study of inverse problems in magnetotellurics has a long history, originating from the seminal works [2–3] that laid the foundations of the theoretical description of the method. Tikhonov proposed the mathematical formulation of the problem of reconstructing the electrophysical properties of the deep layers of the Earth's crust, while Cagniard developed the classical theory of magnetotelluric sounding. These pioneering contributions set the direction for subsequent studies devoted to the development of inversion techniques and the interpretation of geoelectrical data.

A widely adopted approach to solving inverse problems in magnetotellurics is the use of variational–regularization methods, where the problem is reformulated as the minimization of a functional with a regularization term. In this direction, particular attention should be paid to the studies [4], where the coefficient inverse problem for the Helmholtz equation is considered. The author emphasizes the importance of selecting appropriate regularization procedures to ensure the stability of the solution. Similar ideas are further developed in [5], which addresses an identification problem related to integrodifferential Maxwell's equations. Further works [6–9] demonstrate the interdisciplinary significance of inverse problem methods and their application to various areas of mathematical physics.

A significant contribution has also been made in the field of rock conductivity analysis. In [10], a numerical method is proposed for determining the dielectric permittivity from the modulus of the electric field intensity vector. Such studies enhance the interpretation of MT results and improve the physical justification of constructed models. The work [11] provides a review of laboratory measurements of rock conductivity and their interpretation in the context of magnetotelluric data. The author discusses the effects of temperature, pressure, mineral composition, and the presence of fluids on the electrophysical parameters, as well as the consistency between laboratory curves and the results of geophysical inversions. The obtained relationships help refine the interpretation of MT results and strengthen the physical validity of geoelectrical models.

Another important direction is the development of locally iterative methods, where inversion is carried out step by step or for separate regions of the medium. In [12], a quasi-one-dimensional approach is proposed, based on approximating the multidimensional problem with a series of one-dimensional models, each refined at a separate step of the iterative process. This method reduces computational costs and improves stability to noise, particularly in cases with strong geoelectrical contrasts. A similar logic is implemented in [13], where a stepwise algorithm is developed using the integral formulation of the forward problem combined with sequential parameter correction. Such discretization ensures numerical stability and allows adaptation to local features of the subsurface structure, which is especially important for strongly stratified models.

Modern studies increasingly employ machine learning techniques and physics-informed neural network algorithms. In [14], a deep learning inversion method incorporating physical equations is proposed for magnetotelluric data. In [15], a neural-network-based forward modeling algorithm is developed and applied to the inversion problem. These approaches significantly accelerate computations and provide robustness in processing large-scale datasets.

The present work is devoted to the numerical solution of the coefficient inverse problem of magnetotelluric sounding. For the numerical simulations, second-order finite-difference schemes are employed, while the inverse problem is formulated as an optimization problem and solved using the Landweber gradient method. The novelty of the present study lies in proposing a full coefficient formulation of the two-dimensional inverse problem of magnetotelluric sounding for the Helmholtz equation with a complex-valued conductivity $\sigma(y,z)$, which makes it possible to consistently describe both vertical and lateral heterogeneities of the geoelectrical medium, going beyond the classical Tikhonov–Cagniard model. Within this formulation, the variation of the misfit functional is derived rigorously, and the corresponding adjoint problem adapted to complex coefficients is constructed, which has not been presented in existing MT inversion studies. Furthermore, the Landweber method is employed directly to solve the two-dimensional coefficient inverse problem, providing a stable alternative to commonly used magnetotelluric inversion approaches such as Gauss–Newton–type methods. The proposed methodology is demonstrated through an original numerical experiment involving a two-dimensional Gaussian anomaly and the computation of the complex impedance, confirming its effectiveness.

Materials and Methods

Problem Formulation. This article investigates the two-dimensional inverse problem of magnetotelluric sounding under E-polarization of the electromagnetic field. The primary objective is to reconstruct the spatial distribution of electrical conductivity based on the known impedance values measured at the Earth's surface. The direct problem is formulated through the Helmholtz equation with a complex coefficient and the corresponding boundary conditions in a rectangular area (Figure 1). The solution of the inverse problem is carried out using a gradient-based method, formulated as the minimization of a functional.

We now turn to the mathematical formulation of the two-dimensional inverse magnetotelluric (MT) sounding problem.

$$u_{yy} + u_{zz} + k(y,z)u = 0, \qquad -l < y < l, 0 < z < H, \tag{1}$$
$$u(-l,z) = u(l,z) = g(z), \qquad 0 \le z \le H, \tag{2}$$
$$u_z(y,0) + \gamma_0 u(y,0) = \Gamma_0, \qquad -l < y < l, \tag{3}$$
$$u_z(y,H) - \gamma_H u(y,H) = 0, \qquad -l < y < l. \tag{4}$$

where $k(y,z) = i\omega\mu\sigma(y,z)$.

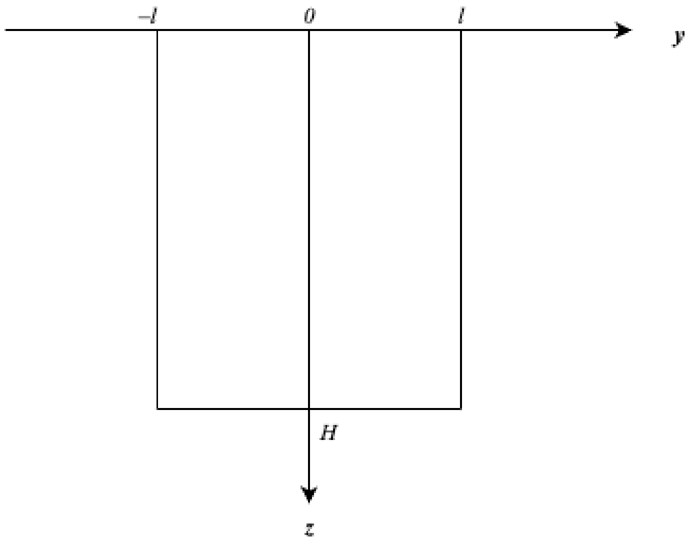


Figure 1 – Domain for the Helmholtz equation

Generalized solution of the two-dimensional MT sounding problem. In problems where the coefficients depend on several variables and complex parameters are involved, the solution may not necessarily possess classical second derivatives. In such cases, the concept of a *generalized solution* in Sobolev spaces is employed [16-17].

Definition 1. A function $u \in L_2(\Omega)$ is called a generalized solution of the direct problem (1)–(4) if for any test function $\varphi \in H^2(\Omega)$ satisfying the following conditions:

$$\varphi(-l,z) = \varphi(l,z) = 0, \tag{5}$$
$$\varphi_z(y,H) - \gamma_H \varphi(y,H) = 0, \tag{6}$$
$$\varphi_z(y,0) + \gamma_0 \varphi(y,0) = 0, \tag{7}$$

the following integral identity holds

$$\int_{-l}^{l}\int_{0}^{H} u\big(\varphi_{yy} + \varphi_{zz} + k\varphi\big)dy\,dz - \Gamma_0 \int_{-l}^{l} \varphi(y,0)dy + \\ + \int_{0}^{H} (\varphi(-l,z) - \varphi(l,z))g(z)dz = 0 \tag{8}$$

By means of Theorem 1, the existence of a generalized solution to the two-dimensional problem (1)–(4) is established, and the corresponding a priori estimate is derived. Similar statements can be found in studies [16-17].

Theorem 1. Let $g \in L^2(0,H), \Gamma_0 \in C,$ the coefficient $k(y,z) \in L^\infty(\Omega)$, and the parameters $\gamma_0, \gamma_H \in C$. Then there exists a unique generalized solution $u \in L^2(\Omega)$, satisfying the weak formulation (8), and this solution satisfies the following a priori estimate:

$$\|u\|_{L^2(\Omega)} \leq C\left(\|g\|_{L^2(0,H)} + |\Gamma_0|\right),$$

where the constant $C > 0$ depends only on the dimensions of the domain $\Omega = (-l{:}\,l) \times (0;H)$ norm $\|k\|_{L^\infty(\Omega)}$, and the coefficients γ_0, γ_H.

Formulation of the direct and inverse problems.

In the direct problem (1)–(4), the objective is to determine the function $u(y,z)$ for given $k(y,z)$ and boundary data $g(z)$. In the inverse problem, the goal is to reconstruct the coefficient $k(y,z)$ using additional observational information.

$$Z_0(y,\omega) = i\omega\mu \cdot \frac{u(y,0)}{u_z(y,0)}. \tag{9}$$

The direct problem, defined by equations (1)–(4), describes the distribution of the complex electric field $u(y,z)$ in a geoelectrical medium for a fixed parameter $k(y,z)$ associated with electrical conductivity. It is well known that $k(y,z)$ is directly determined by the conductivity distribution, and its recovery relies on experimentally obtained data.

In magnetotelluric sounding, the primary observable quantity is the impedance, which represents the ratio of the electric and magnetic field components. In the two-dimensional model with E-polarization, the impedance takes the form (9) and serves as the only boundary characteristic directly related to the conductivity coefficient $\sigma(y,z)$. Since the impedance contains essential information about both vertical and lateral variations in electrical conductivity and is widely used in the interpretation of MT data, minimizing the misfit between the observed impedance and the impedance reconstructed by the model is a physically justified and commonly accepted approach. This choice of the functional ensures proper agreement between the model and the physically measurable quantities and makes the inversion problem consistent with classical MT interpretation practices.

The main objective of the inverse problem is to reconstruct the parameter $k(y,z)$ from the known impedance values $Z_0(y,\omega)$ measured at the boundary $z = 0$. To achieve this goal, a misfit functional is introduced, the minimization of which formulates the inverse problem as an optimization task, typically solved with the aid of regularization techniques.

$$\mathcal{J}(k) = \sum_{\omega=\omega_0}^{\omega_N} \int_{-l}^{l} \left(Z_0(y,\omega) \cdot u_z(y,0;k) - i\omega\mu \cdot u(y,0;k)\right)^2 dy \tag{10}$$

where $Z_0(y,\omega)$ denotes the impedance computed from the observational data, and $u(y,z;k)$ corresponds to the solution of the direct problem for a fixed distribution of the parameter k.

We define the variation of the functional [6, 16-17]:

$$\mathcal{J}(k+\delta k) - \mathcal{J}(k) = \sum_{\omega} \int_{-l}^{l} \left(Z_0(y,\omega) \cdot u_z(y,0;k+\delta k) - i\omega\mu \cdot u(y,0;k+\delta k)\right)^2 dy -$$
$$- \sum_{\omega} \int_{-l}^{l} \left(Z_0(y,\omega) \cdot u_z(y,0;k) - i\omega\mu \cdot u(y,0;k)\right)^2 dy =$$

$$= \sum_{\omega} \int_{-l}^{l} \left[Z_0(y,\omega) \cdot \left(u_z(y,0;k+\delta k) - u_z(y,0;k) \right) - i\omega\mu \cdot \left(u(y,0;k+\delta k) - u(y,0;k) \right) \right] \times$$
$$\times \left[Z_0(y,\omega) \cdot \left(u_z(y,0;k+\delta k) + u_z(y,0;k) \right) - i\omega\mu \cdot \left(u(y,0;k+\delta k) + u(y,0;k) \right) \right] dy$$

Let's introduce a substitution:

$$u(y,z;k+\delta k) = \tilde{u}$$
$$u(y,z;k) = u$$
$$\tilde{u} - u = \delta u$$
$$u = \delta u + \tilde{u}$$

With this substitution, we arrive at the following representation:

$$\sum_{\omega} \int_{-l}^{l} [Z_0(y,\omega) \cdot \delta u_z(y,0) - i\omega\mu \cdot \delta u(y,0)] \times$$
$$\times 2[Z_0(y,\omega) \cdot u_z(y,0) - i\omega\mu \cdot u(y,0)] dy + O(\|\delta u\|)$$

Let us formulate the perturbed problem:

$$\tilde{u}_{yy} + \tilde{u}_{zz} + (k+\delta k)\tilde{u} = 0 \quad (11)$$
$$\tilde{u}(-l,z) = \tilde{u}(l,z) = g(z) \quad (12)$$
$$\tilde{u}_z(y,0) + \gamma_0 \tilde{u}(y,0) = \Gamma_0 \quad (13)$$
$$\tilde{u}_z(y,H) - \gamma_H \tilde{u}(y,H) = 0 \quad (14)$$

Subtracting problem (1)–(4) from problem (11)–(14), we obtain the problem for δu

$$\delta u_{yy} + \delta u_{zz} + k\delta u + \delta k u = 0 \quad (15)$$
$$\delta u(-l,z) = \delta u(l,z) = 0 \quad (16)$$
$$\delta u_z(y,0) + \gamma_0 \delta u(y,0) = 0 \quad (17)$$
$$\delta u_z(y,H) - \gamma_H \delta u(y,H) = 0 \quad (18)$$

Starting from (15), we take the identity equal to zero and, after integrating by parts, arrive at the following expression:

$$0 = \sum_{\omega} \int_{-l}^{l} \int_{0}^{H} \left[\delta u_{yy} + \delta u_{zz} + k \cdot \delta u + \delta k \cdot u \right] \cdot \psi(y,z) dz\, dy =$$
$$= \sum_{\omega} \left(\int_{0}^{H} \left[\delta u_y \cdot \psi|_{-l}^{l} - \delta u \cdot \psi_y|_{-l}^{l} + \int_{-l}^{l} \psi_{yy} \delta u dy \right] dy \right.$$
$$+ \int_{-l}^{l} \left[\delta u_z \cdot \psi|_{0}^{H} - \delta u \cdot \psi_z|_{0}^{H} + \int_{0}^{H} \psi_{zz} \delta u dz \right] dy +$$
$$\left. + \int_{-l}^{l} \int_{0}^{H} (k \cdot \delta u \cdot \psi + \delta k \cdot u \cdot \psi) dz\, dy \right) = \sum_{\omega} \left(\int_{-l}^{l} \int_{0}^{H} \left[\psi_{yy} + \psi_{zz} + k \cdot \psi \right] \delta u dz\, dy + \right.$$
$$+ \int_{-l}^{l} \int_{0}^{H} \delta k \cdot \psi \cdot u dz\, dy + \int_{0}^{H} \left[\delta u_y(l,z) \cdot \psi(l,z) - \delta u_y(-l,z) \cdot \psi(-l,z) \right] dz -$$

$$-\int_0^H [\delta u(l,z)\cdot\psi_z(l,z) - \delta u(-l,z)\cdot\psi_z(-l,z)]dz +$$
$$+\int_{-l}^{l}[\delta u_z(y,H)\cdot\psi(y,H) - \delta u_z(y,0)\cdot\psi(y,0) - \delta u(y,H)\cdot\psi_z(y,H)$$
$$+\delta u(y,0)\cdot\psi_z(y,0)]dy) =$$

Based on (16), (17), (18), we obtain:

$$=\sum_{\omega}\Big(\int_{-l}^{l}\int_0^H\big[\psi_{yy}+\psi_{zz}+k\cdot\psi\big]\delta u dz\,dy + \int_{-l}^{l}\int_0^H \delta k\cdot\psi\cdot u dz\,dy +$$
$$+\int_0^H\big[\delta u_y(l,z)\cdot\psi(l,z) - \delta u_y(-l,z)\cdot\psi(-l,z)\big]dz +$$
$$+\int_{-l}^{l}\big[\big(-\psi_z(y,H)+\gamma_H\psi(y,H)\big)\delta u(y,H) + \big(\psi_z(y,0)-\gamma_0\psi(y,0)\big)\delta u(y,0)\big]dy\Big).$$

Alternatively, the variation of the functional can be expressed directly as:

$$\mathcal{J}(k+\delta k)-\mathcal{J}(k) = \langle\delta_k, \mathcal{J}_k'\rangle =$$
$$\sum_{\omega}\int_{-l}^{l}\big[Z_0(y,\omega)\cdot\delta u_z(y,0) - i\omega\mu\cdot\delta u(y,0)\cdot 2[Z_0(y,\omega)\cdot u_z(y,0) - i\omega\mu\cdot u(y,0)]\big]dy.$$

Accordingly, the adjoint problem can be formulated as follows

$$\psi_{yy}+\psi_{zz}+k\cdot\psi = 0 \tag{19}$$
$$\psi(-l,z)=\psi(l,z)=0 \tag{20}$$
$$\psi_z(y,H)+\gamma_H\psi(y,H)=0 \tag{21}$$
$$\psi_z(y,0)-\gamma_0\psi(y,0) = 2(\gamma_0\cdot Z_0(y,\omega)+i\omega\mu)\times \times\big(Z_0(y,\omega)\cdot u_z(y,0) - i\omega\mu\cdot u(y,0)\big) \tag{22}$$

This yields the following expression:

$$0=\sum_{\omega}\int_{-l}^{l}\int_0^H \delta k\cdot\psi\cdot u dz\,dy + \int_{-l}^{l}\delta u(y,0)\cdot\big(\psi_z(y,0)-\gamma_0\psi(y,0)\big)dy$$

Accordingly, the gradient of the functional can be expressed in the following form:

$$\mathcal{J}'(k)=\sum_{\omega}\psi(y,z;\omega)\cdot u(y,z;\omega) \tag{23}$$

Here, $\psi(y,z;\omega)$ represents the solution of the adjoint problem (19)–(22).

Algorithm for Solving the Inverse Problem Based on the Landweber Method. The inverse problem, in its numerical form, is formulated as an optimization task, the goal of which is to minimize a specific functional. The efficiency achieved and the stability of convergence are directly related to the choice of the optimization method[6,7,9,17].

Landweber Iterative Algorithm.

1. Initialization.

Choose an initial approximation for the unknown coefficient, $k^{(0)}$;

2. Solution of the direct problem.

Assuming that the current approximation $k^{(n)}$ is known, solve the discrete direct problem (1)–(4) using a finite-difference scheme. In the interior nodes, the Helmholtz equation is approximated as

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h_y^2} + \\ + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h_z^2} + k_{i,j} \cdot u_{i,j} = 0 \tag{24}$$

subject to the boundary conditions

$$u_{0,j} = u_{N_y,j} = g(z_i) \tag{25}$$

$$\left(\frac{-3}{2h_z} + \gamma_0\right) \cdot \\ \cdot u_{i,0} + \frac{2}{h_z} \cdot u_{i,1} - \frac{1}{2h_z} \cdot u_{i,2} = \Gamma_0 \tag{26}$$

$$\frac{1}{2h_z} \cdot u_{i,N_z-2} - \frac{2}{h_z} \cdot u_{i,N_z-1} + \left(\frac{3}{2h_z} - \gamma_H\right) \\ \cdot u_{i,N_z} = 0 \tag{27}$$

3. Evaluation of the misfit functional.

Compute the value of the functional $\mathcal{J}\left(k^{(n)}\right)$ according to formula (10);

4. Solution of the adjoint problem.

If the functional value is not sufficiently small, solve the adjoint problem (19)–(22). The discrete adjoint equation in the interior of the domain has the form

$$\frac{\psi_{i+1,j} - 2\psi_{i,j} + \psi_{i-1,j}}{h_y^2} + \\ + \frac{\psi_{i,j+1} - 2\psi_{i,j} + \psi_{i,j-1}}{h_z^2} + k_{i,j} \cdot \psi_{i,j} = 0 \tag{28}$$

with boundary conditions

$$\psi_{0,j} = \psi_{N_y,j} = 0 \tag{29}$$

$$\frac{1}{2h_z} \cdot \psi_{i,N_z-2} - \frac{2}{h_z} \cdot \psi_{i,N_z-1} + \\ + \left(\frac{3}{2h_z} + \gamma_H\right) \cdot \psi_{i,N_z} = 0 \tag{30}$$

$$-\left(\frac{3}{2h_z} + \gamma_0\right) \cdot \psi_{i,0} + \frac{2}{h_z} \cdot \psi_{i,1} - \frac{1}{2h_z} \cdot \psi_{i,2} \\ = \theta_j \tag{31}$$

where

$$\theta_j = \theta(y_j) = 2\left(\gamma_0 \cdot Z_0(y_j, \omega) + i\omega\mu\right) \cdot \\ \cdot \left(Z_0(y_j, \omega) \cdot u_z(y_j, 0) - i\omega\mu \cdot u(y_j, 0)\right)$$

5. Gradient computation.

Compute the gradient of the functional

$$\mathcal{J}' k^{(n)} = \sum_{\omega} \psi_{i,j}(\omega) \cdot u_{i,j}(\omega)$$

6. Update step.

Update the coefficient using the Landweber iteration:

$$k^{(n+1)} = k^{(n)} - \alpha \mathcal{J}' k^{(n)}$$

and return to step 2. Where $\alpha \in (0, \|A\|^{-2})$ is the step-size (descent) parameter [16].

7. Stopping criterion.

The iterative process is terminated either when the functional value stabilizes or, in the presence of noisy data, when a discrepancy-type stopping rule is satisfied $\mathcal{J}\left(k^{(n)}\right) < \varepsilon^2$. In particular, an early stopping criterion is used, which acts as an implicit regularization mechanism for the ill-posed inverse problem [18].

Here, the Landweber method serves not only as an optimization technique but also as an iterative regularization method, where stability with respect to data perturbations is achieved by an appropriate choice of the step size α and by early termination of the iterations.

Results and Discussion

To assess the performance of the proposed numerical algorithm, we consider a two-dimensional conductivity model $\sigma(y, z)$ consisting of a background distribution and a localized Gaussian anomaly. The background is defined by a function $\sigma^N(z)$ that depends only on depth z, whereas the anomaly is introduced in the central part of the computational domain and is localized with respect to both the y and z coordinates.

In the course of numerical simulation, a complete two-dimensional system of linear algebraic equations (SLAE) was constructed, arising from the finite-difference approximation of the Helmholtz equation with the variable complex coefficient $k(y,z) = i\omega\mu\sigma(y,z)$. The resulting system is large-scale and sparse, which renders the use of classical direct solvers inefficient. Therefore, to obtain the numerical solution, a sparse-matrix method was employed, ensuring a significant reduction in computational cost and improved stability of the algorithm when modeling realistic geoelectrical media.

Parameters of the Numerical Experiment. Within the framework of the numerical experiment, a two-dimensional model was considered that includes a Gaussian conductivity anomaly localized in a bounded region. The geometric parameters of the computational domain are specified as follows: the half-width along the horizontal axis is $l = 1$, and the depth of the domain is $H = 1$. The simulation frequency range was chosen as $f \in [1,10]$ with a discretization step of 0.2, while the angular frequency is defined by the relation $\omega = 2\pi f$.

To prescribe the anomalous conductivity, a Gaussian function is employed:

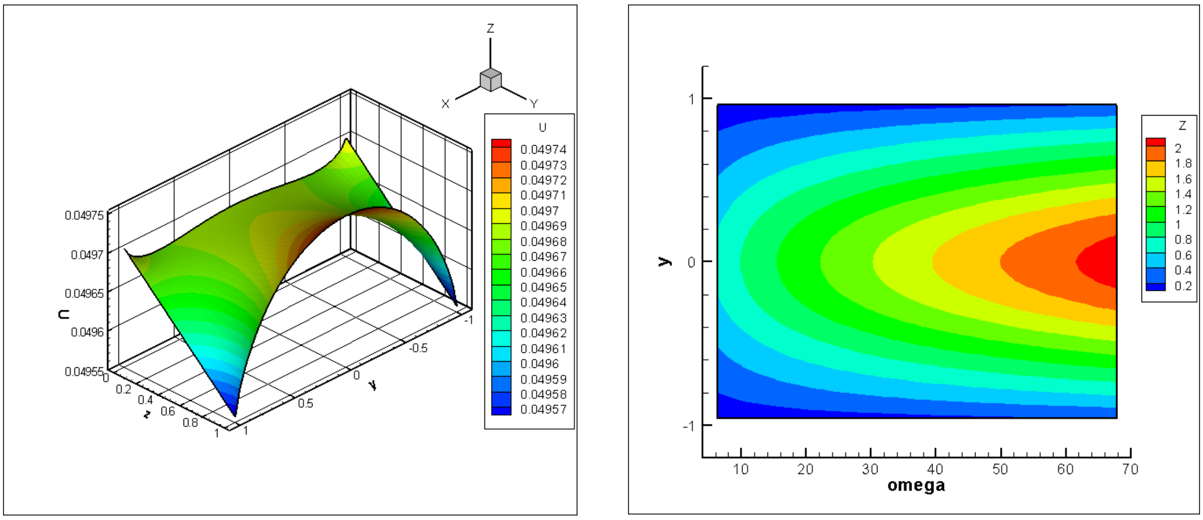
$$\sigma_{ex}(y,z) = 0.1 \cdot e^{-\left((y/0.5)^2+\left((z-0.6)/0.2\right)^2\right)}$$

The air conductivity in the model is set to $\sigma_0 = 0.01$, whereas at the lower boundary of the computational domain the value $\sigma_H = 0.1$ is used. The magnetic permeability of the medium is assumed to be constant throughout the entire domain and is taken as $\mu = 0.08\pi$.

In Figure 2a, the real part of the field, $a)Re\big(u(y,z)\big)$, exhibits smooth spatial variations: the values reach their maximum in the central part of the domain and gradually decrease toward the boundaries. This distribution reflects the influence of the central conductive anomaly and the symmetry of the geoelectrical structure with respect to the y-axis.

In Figure 2b, the real part of the impedance, $b)Re\big(Z_0(y,\omega)\big)$, shows a pronounced maximum in the central zone (near $y = 0$) and decreases toward the edges ($y = \pm 1$). As the frequency increases, an overall reduction of $Re\big(Z_0(y,\omega)\big)$, is observed, which corresponds to the shallower penetration depth of the electromagnetic field and the reduced contribution of deep conductive structures.

Both presented quantities – $a)Re\big(u(y,z)\big)$, and $Re\big(Z_0(y,\omega)\big)$ – behave consistently and serve as the basis for further impedance analysis and reconstruction of the medium structure. The imaginary parts of the field and impedance are not considered in this section, since the focus is placed on amplitude (energy) characteristics; their analysis is beyond the scope of the current illustration.

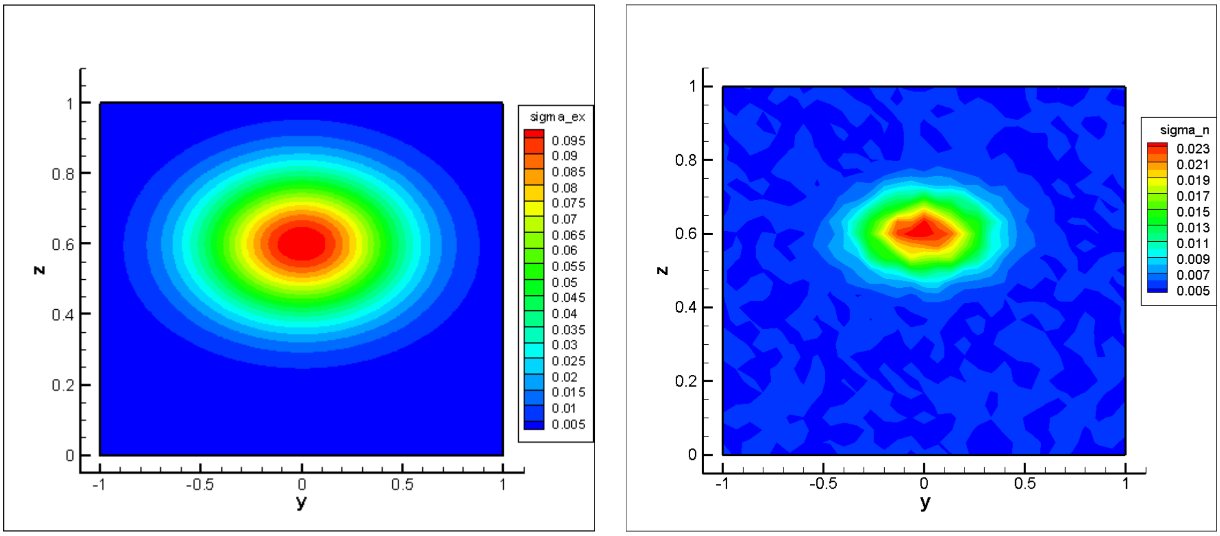


$a)Re\big(u(y,z)\big)$ $b)Re\big(Z_0(y,\omega)\big)$

Figure 2 – Numerical solution of the direct problem with the specified model parameters: a) distribution of the electromagnetic field $u(y,z)$; b) impedance response

Figure 3 presents the results of the numerical solution of the two-dimensional inverse magnetotelluric sounding problem obtained using the Landweber method. The exact conductivity distribution $\sigma_{ex}(y,z)$ (Figure 3a) is compared with the approximate solution $\sigma_n(y,z)$ (Figure 3b), computed after a finite number of iterations.

The exact distribution is characterized by a smooth and symmetric structure with a maximum at the center and gradual decay toward the boundaries of the domain. The reconstructed solution generally reproduces the shape of the original model: the central maximum coincides in position, and the main contours of the anomaly are preserved. The largest discrepancies between the exact and approximate solutions are observed in the peripheral regions, which can be explained by the higher sensitivity to data errors and iterative inaccuracies.



$a)\sigma_{ex}(y,z)$ в 2D $b)\sigma_n(y,z)$ в 2D

Figure 3 – Graphical comparison of the exact and reconstructed conductivity

Figure 4 illustrates the results of the numerical solution of the inverse magnetotelluric sounding problem obtained with the Landweber method. Shown are the convergence curves of the functional $\mathcal{J}(k)$, which exhibit a monotonic decrease throughout the iterative process.

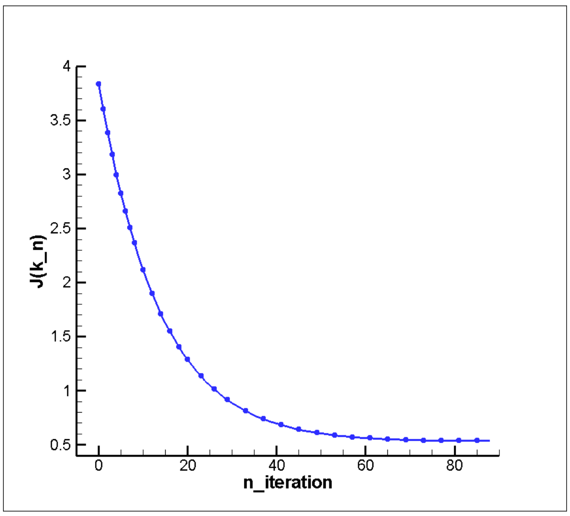


Figure 4 – Graph of the functional decrease using the Landweber method

As a test case, an exponential function was employed, whose reconstruction is traditionally considered challenging in inverse problems. Future work will focus on applying the methodology to real field data and on extending the approach to more complex types of anomalies. The present study is primarily aimed at demonstrating the methodological framework for solving inverse problems in magnetotelluric sounding.

It should be noted that in the present study the numerical experiments were performed using synthetic data without the addition of artificial noise. Even in this case, the results demonstrate that the Landweber method does not provide a complete reconstruction of the conductivity coefficient and leads to a smoothing of the solution, especially in the peripheral regions of the computational domain. Such behavior is typical for gradient-based iterative methods when solving ill-posed coefficient inverse problems and reflects their regularizing nature.

In this regard, the additional introduction of noise into the input data is not fundamentally necessary to reveal the limitations of the method, since the main effects related to stability and incomplete reconstruction already appear for idealized data. The present work is focused on demonstrating the methodological framework for solving the two-dimensional coefficient inverse problem of magnetotelluric sounding and on analyzing the behavior of the Landweber algorithm. A more detailed investigation of the influence of measurement noise, regularization parameters, and stopping criteria is planned for future studies.

Conclusion

This study addressed the two-dimensional inverse problem of magnetotelluric sounding. The work is of a review-applied nature and was aimed at developing a direct numerical model of the electromagnetic response of a geoelectrical medium. The modeling framework is based on the Helmholtz equation with a complex coefficient, in which conductivity varies both with depth and along the transverse coordinate.

A synthetic model was designed for the numerical experiments, consisting of a smooth background conductivity and a localized anomaly. Simulations of the electromagnetic field and surface impedance revealed the strong sensitivity of the response to subsurface heterogeneities. The observed field symmetry and the occurrence of local maxima in the impedance distribution validate the correctness of the implemented algorithm.

The inverse problem of reconstructing the conductivity distribution from synthetic data was also examined. The numerical results demonstrated that even with limited information, a high level of reconstruction accuracy can be achieved. The misfit functional attained its minimum after a relatively small number of iterations, and the approximate solution closely matched the original model.

Overall, the findings confirm that the proposed numerical scheme and inversion algorithm are effective both for direct modeling and for the stable solution of inverse magnetotelluric sounding problems, highlighting their potential as practical tools for geophysical interpretation. Quantitative error analysis and comparison with alternative inversion methods will be the subject of future work.

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