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On the inverse problem of identifying the source term in a pseudoparabolic equation with a final time overdetermination condition

Abstract. In this paper, we consider the inverse problem for a linear pseudoparabolic equation describing the temperature distribution taking into account external forces that depend only on the spatial variable. The classical solution to the inverse problem under consideration satisfies the usual pseudoparabolic equation, initial and nonlocal boundary conditions, and a final additional condition. The issues of existence and uniqueness of the solution to the inverse problem are the subject of study in the work presented by the author. As the main result, theorems on the existence and uniqueness of the classical solution to the problem under study are formulated and rigorously proven. These theorems are completely proven in a mathematically rigorous language using the method of separation of variables. In the course of the proof, a system of orthogonal and biorthogonal basis functions of a special type was chosen in accordance with the nonlocal boundary conditions. First of all, to prove the theorem on the existence of a solution, an analytical formula for the solution was derived in the form of a series in the system of these functions, their uniform convergence was analyzed according to the Weierstrass theorem, and the convergence to the classical solution of the inverse problem under consideration was investigated. The proof of the theorem on uniqueness was carried out by the method of the opposite assumption.
Keywords: inverse problem, pseudoparabolic equation, existence of solution, uniqueness of solution, nonlocal boundary condition.

Introduction

In this paper, we study the inverse problem for a linear pseudoparabolic equation that includes an additional term accounting for the effect of external forces depending only on the spatial variable, with the goal of determining the temperature. The classical solution of this inverse problem satisfies the standard pseudoparabolic equation, along with the initial and nonlocal boundary conditions, as well as the final additional condition. The primary objective of this work is to prove the existence, uniqueness and stability of the solution to the inverse problem.

The nonlocal boundary conditions considered in the present study were first introduced in the work of N.I. Ionkin [1]. Later, A.A. Samarskii [2] investigated the formulation of differential equations with nonstandard boundary conditions. In accordance with these works, boundary value problems with the nonlocal boundary condition (3) became known as the *Ionkin–Samarskii problem*. However, historically, problems with nonlocal boundary conditions can be traced back to the work of V.A. Steklov [3].

In general, pseudoparabolic equations are employed to describe important physical processes such as hydrodynamics, filtration theory, continuum mechanics, heat conduction in two-temperature systems, dispersive flows, viscous flows in materials with memory, and others. As an example, one may mention the Kelvin–Voigt (Navier–Stokes–Voigt) system of equations. The formulation of equation (1) from physical laws and its mathematical modeling can be found in the works [4-8].

There are very few studies devoted to the inverse problem of determining the right-hand side of a given equation that depends on the spatial variable, or of reconstructing the right-hand side itself. Under suitable assumptions on the given data, authors prove existence and uniqueness of a classical solution (u, f) to the corresponding inverse problem [9]. The authors of [10] successfully obtained a stable and precise reconstruction of the unknown spatial source term using a combination of finite-difference methods and Tikhonov regularization. Theoretical guarantees: existence, uniqueness and stability of solution, and numerical solution (B-spline + Tikhonov regularization + nonlinear least-squares) are established by Huntul M.J., Khompysh Kh., Shazyndayeva M.K., Iqbal M.K. [11]. In [12], the authors demonstrated both theoretically and numerically the existence of solutions using fractional Landweber iterative regularization. Author [13] have studied existence and uniqueness of a classical solution to appropriate inverse problems, under suitable assumptions on the initial/boundary data. See also [16] and the references therein.

In this work, the research methodology is purely theoretical. The main objective is to establish the existence, uniqueness and stability of a classical solution to the inverse problem (1)–(4). The main tool is the Fourier method in combination with orthogonal and biorthogonal systems of functions adapted to the nonlocal boundary conditions. The proof of existence relies on the derivation of an analytical formula for the solution in the form of uniformly convergent series, justified by the Weierstrass theorem. The proof of uniqueness is based on the method of contradiction.

Materials and Methods

In this section, we present the formulation of the inverse problem together with the methods used for its analysis.

Problem statement. Let us consider the inverse problem of determining the pair of functions $(u(x,t), f(x))$ in the rectangle $Q_T = \{(x,t): x \in (0,1), t \in (0,T), T < \infty\}$. The inverse problem is to find $u(x,t)$ and $f(x)$ satisfying the pseudoparabolic equation

$$u_t - u_{xx} - u_{xxt} = f(x), \quad (x,t) \in Q_T \tag{1}$$

the initial condition

$$u(x,0) = \varphi(x), \quad x \in [0,1] \tag{2}$$

the non-local periodic boundary conditions

$$\begin{gathered} u(1,t) = 0, \\ u_x(0,t) = u_x(1,t), \;\; t \in [0,T], \end{gathered} \tag{3}$$

and the additional final overdetermination condition

$$u(x,T) = \psi(x), \quad x \in [0,1], \tag{4}$$

where $\varphi(x)$ and $\psi(x)$ are given functions, whereas $f(x)$ and $u(x,t)$ are the unknowns to be determined.

Results and Discussion

Firstly, let's give the definition of a solution to the inverse problem (1)–(4).

Definition 1. A solution of the inverse problem (1)–(4) is defined as a pair of functions $(u(x,t), f(x))$ belonging respectively to the spaces $C^{2,1}_{x,t}(Q_T) \cap C^{1,0}_{x,t}\left(\overline{Q_T}\right)$ and $\mathrm{C}([0,1])$, which satisfy the pseudoparabolic equation (1) in the rectangle Q_T, the initial condition (2) and the final condition (4) on the interval $[0,1]$, as well as the boundary condition (3) on the interval $[0,T]$.

Existence of the solution. In this section, the existence of a solution to the inverse problem (1)–(4) is established, and the following result is obtained.

Theorem 1. Let $\varphi(x), \psi(x) \in C^4([0,1])$ be functions satisfying the compatibility conditions

$$\varphi(1) = 0, \quad \varphi'(1) = \varphi'(0), \tag{5}$$

$$\psi(1) = 0, \quad \psi'(0) = \psi'(1), \tag{6}$$

$$\varphi''(0) = \varphi''(1), \quad \psi''(0) = \psi''(1) \tag{7}$$

$$\varphi'''(0) = \varphi'''(1), \quad \psi'''(0) = \psi'''(1) \tag{8}$$

Then there exists a classical solution $(u(x,t), f(x))$, belonging respectively to the spaces $C^{2,1}_{x,t}(Q_T)$ and $\mathrm{C}([0,1])$, of the inverse problem (1)–(4).

Proof. The proof of Theorem 1 is based on the Fourier method.

We seek the solution of the inverse problem (1)–(4) by expanding in series with respect to the special system of basis functions

$$2(1-x),\quad \{4(1-x)\cos 2\pi nx\}_{n=1}^{\infty},\quad \{4\sin 2\pi nx\}_{n=1}^{\infty} \tag{9}$$

The system (9) was studied in [14,15] and shown to be complete in $L^2(0,1)$ and to form a Riesz basis. However, the system (9) is not orthogonal; for it the following biorthogonal system was given in [15]

$$1,\quad \{\cos 2\pi nx\}_{n=1}^{\infty},\quad \{x\sin 2\pi nx\}_{n=1}^{\infty}. \tag{10}$$

The system (10) is used both to prove the uniqueness of the solution of the inverse problem (1)–(4) and to determine the Fourier coefficients that appear when the solution is expanded with respect to the system (9). Hence, the solution pair $u(x,t)$ and $f(x)$ of the inverse problem (1)–(4) can be written in the series form

$$u(x,t)=2u_0(t)(1-x)+\sum_{n=1}^{\infty}4\,u_{1n}(t)(1-x)\cos 2\pi nx+\sum_{n=1}^{\infty}4u_{2n}(t)\sin 2\pi nx \tag{11}$$

$$f(x)=2f_0(1-x)+\sum_{n=1}^{\infty}4f_{1n}(1-x)\cos 2\pi nx+\sum_{n=1}^{\infty}4f_{2n}\sin 2\pi nx \tag{12}$$

where $u_0(t),\ u_{1n}(t),\ u_{2n}(t)$ and the constants f_0, f_{1n}, f_{2n} are the unknown Fourier coefficients. Formally assuming that the series for $u(x,t)$ and $f(x)$ can be differentiated termwise, we substitute these series into equation (1).

Note that differentiating the series (11)–(12) termwise yields the following expressions for $u_t(x,t), u_{xx}(x,t)$ and $u_{xxt}(x,t)$

$$u_t(x,t)=2u_0'(t)(1-x)+\sum_{n=1}^{\infty}4u_{1n}'(t)(1-x)\cos 2\pi nx+\sum_{n=1}^{\infty}4u_{2n}'(t)\sin 2\pi nx, \tag{13}$$

$$u_{xx}(x,t)=\sum_{n=1}^{\infty}4u_{1n}(t)\left(4\pi n\sin 2\pi nx-4\pi^2n^2(1-x)\cos 2\pi nx\right)+\sum_{n=1}^{\infty}4u_{2n}(t)\left(-4\pi^2n^2\sin 2\pi nx\right), \tag{14}$$

$$u_{xxt}(x,t)=\sum_{n=1}^{\infty}4u_{1n}'(t)\left(4\pi n\sin 2\pi nx-4\pi^2n^2(1-x)\cos 2\pi nx\right)+\sum_{n=1}^{\infty}4u_{2n}'(t)\left(-4\pi^2n^2\sin 2\pi nx\right). \tag{15}$$

Plugging (13)–(15) into (1) yields

$$\begin{aligned}&2u_0'(t)(1-x)+\sum_{n=1}^{\infty}4\,u_{1n}'(t)\left((1-x)\cos 2\pi nx-4\pi n\sin 2\pi nx-4\pi^2n^2(1-x)\cos 2\pi nx\right)-\\&-\sum_{n=1}^{\infty}4u_{1n}(t)\left(4\pi n\sin 2\pi nx-4\pi^2n^2(1-x)\cos 2\pi nx\right)+\sum_{n=1}^{\infty}4u_{2n}'(t)\left(1+4\pi^2n^2\right)\sin 2\pi nx+\\&+\sum_{n=1}^{\infty}16\pi^2n^2u_{2n}(t)\sin 2\pi nx=2f_0(1-x)+\sum_{n=1}^{\infty}4f_{1n}(1-x)\cos 2\pi nx+\sum_{n=1}^{\infty}4f_{2n}\sin 2\pi nx\end{aligned} \tag{16}$$

Also, imposing the initial and final conditions (2) and (4) on the expansions (11) and (12) yields the relations

$$u_0(0)=\varphi_0;\ u_{1n}(0)=\varphi_{1n};\ u_{2n}(0)=\varphi_{2n}; \quad (17)$$

$$u_0(T)=\psi_0;\ u_{1n}(T)=\psi_{1n};\ u_{2n}(T)=\psi_{2n}. \quad (18)$$

where $\varphi_0, \varphi_{1n}, \varphi_{2n}$ and $\psi_0, \psi_{1n}, \psi_{2n}$ are, respectively, the Fourier coefficients of $\varphi(x)$ and $\psi(x)$ with respect to the system (10). They are given by

$$\varphi_0=\int_0^1 \varphi(x)dx,$$
$$\varphi_{1n}=\int_0^1 \varphi(x)\cos 2\pi nx dx, \quad (19)$$
$$\varphi_{2n}=\int_0^1 \varphi(x)x\sin 2\pi nx dx,$$

$$\psi_0=\int_0^1 \psi(x)dx,$$
$$\psi_{1n}=\int_0^1 \psi(x)\cos 2\pi nx dx, \quad (20)$$
$$\psi_{2n}=\int_0^1 \psi(x)x\sin 2\pi nx dx.$$

Thus, from equation (16) and the conditions (17), (18) we obtain the following systems of ordinary differential equations for the unknown coefficient functions $u_0(t), u_{1n}(t), u_{2n}(t)$ and the constants f_0, f_{1n}, f_{2n}

$$\begin{cases} u_0'(t)=f_0, \\ u_0(0)=\varphi_0; \quad u_0(T)=\psi_0. \end{cases} \quad (21)$$

$$\begin{cases} u_{1n}'(t)+\dfrac{4\pi^2n^2}{1+4\pi^2n^2}u_{1n}(t)=\dfrac{f_{1n}}{1+4\pi^2n^2}, \\ u_{1n}(0)=\varphi_{1n}; \quad u_{1n}(T)=\psi_{1n}. \end{cases} \quad (22)$$

$$\begin{cases} u_{2n}'(t)+\dfrac{4\pi^2n^2}{1+4\pi^2n^2}u_{2n}(t)=\dfrac{f_{2n}}{1+4\pi^2n^2}+\dfrac{4\pi n}{1+4\pi^2n^2}u_{1n}(t)+\dfrac{4\pi n}{1+4\pi^2n^2}u_{1n}'(t), \\ u_{2n}(0)=\varphi_{2n}; \quad u_{2n}(T)=\psi_{2n}. \end{cases} \quad (23)$$

Note that the solution of problem (23) can be found only after solving (22).

Firstly, the solution of (21) is

$$u_0(t)=\frac{\psi_0-\varphi_0}{T}t+\varphi_0,$$
$$f_0=\frac{\psi_0-\varphi_0}{T}; \quad (24)$$

The solution of (22) is obtained explicitly as

$$u_{1n}(t)=\frac{\psi_{1n}-\varphi_{1n}}{e^{-\frac{4\pi^2n^2}{1+4\pi^2n^2}T}-1}\left(e^{-\frac{4\pi^2n^2}{1+4\pi^2n^2}t}-1\right)+\varphi_{1n},$$
$$f_{1n}=4\pi^2n^2\left(\varphi_{1n}-\frac{\psi_{1n}-\varphi_{1n}}{e^{-\frac{4\pi^2n^2}{1+4\pi^2n^2}T}-1}\right) \quad (25)$$

Finally, the solution of (23) can be written in the form

$$u_{2n}(t)=\varphi_{2n}+C_{2n}\left(e^{-\frac{4\pi^2n^2}{1+4\pi^2n^2}t}-1\right)+\frac{4\pi n}{(1+4\pi^2n^2)^2}te^{-\frac{4\pi^2n^2}{1+4\pi^2n^2}t},$$
$$f_{2n}=4\pi^2n^2\left(\varphi_{2n}-C_{2n}\right)-\pi n\left(\varphi_{1n}-\frac{\psi_{1n}-\varphi_{1n}}{e^{-\frac{4\pi^2n^2}{1+4\pi^2n^2}T}-1}\right) \quad (26)$$

where

$$C_{2n} = \frac{1}{e^{-\frac{4\pi^2 n^2}{1+4\pi^2 n^2}T} - 1}\left(\psi_{2n} - \varphi_{2n} - \frac{4\pi n}{(1+4\pi^2 n^2)^2} T e^{-\frac{4\pi^2 n^2}{1+4\pi^2 n^2}T} \right). \tag{27}$$

Now, substituting the functions $u_0(t),\, u_{1n}(t),\, u_{2n}(t)$, defined by $(24)_1$, $(25)_1$, $(26)_1$ into expression (11), and the numbers f_0, f_{1n}, f_2, defined by $(24)_2$, $(25)_2$, $(26)_2$ into expression (12), we obtain the following functions $u(x,t)$ and $f(x)$

$$\begin{aligned} u(x,t) &= \varphi(x) + 2\frac{\psi_0 - \varphi_0}{T}\, t\,(1-x) + \sum_{n=1}^{\infty} 4 \left(\frac{\psi_{1n} - \varphi_{1n}}{e^{-\frac{4\pi^2 n^2}{1+4\pi^2 n^2}T} - 1} \left(e^{-\frac{4\pi^2 n^2}{1+4\pi^2 n^2}t} - 1 \right) + \varphi_{1n} \right)(1-x)\cos 2\pi n x + \\ &+ \sum_{n=1}^{\infty} 4 \left(\varphi_{2n} + C_{2n} \left(e^{-\frac{4\pi^2 n^2}{1+4\pi^2 n^2}t} - 1 \right) + \frac{4\pi n}{(1+4\pi^2 n^2)^2} t e^{-\frac{4\pi^2 n^2}{1+4\pi^2 n^2}t} \right) \sin 2\pi n x \end{aligned} \tag{28}$$

$$\begin{aligned} f(x) &= 2\frac{\psi_0 - \varphi_0}{T}(1-x) + \sum_{n=1}^{\infty} 4\pi^2 n^2 \left(\varphi_{1n} - \frac{\psi_{1n} - \varphi_{1n}}{e^{-\frac{4\pi^2 n^2}{1+4\pi^2 n^2}T} - 1} \right)(1-x)\cos 2\pi n x + \\ &+ \sum_{n=1}^{\infty} 4\pi^2 n^2 \left(\varphi_{2n} - C_{2n} \right) - 4\pi n \left(\varphi_{1n} - \frac{\psi_{1n} - \varphi_{1n}}{e^{-\frac{4\pi^2 n^2}{1+4\pi^2 n^2}T} - 1} \right) \sin 2\pi n x. \end{aligned} \tag{29}$$

Now, we show that the solutions (28), (29) satisfy equation (1). For this purpose, substituting expressions (19)–(20) into the series (28), (29), and applying the formula of integration by parts together with the compatibility conditions (5)–(8), we can rewrite them as follows:

$$\begin{aligned} u(x,t) &= \varphi(x) + 2\frac{\psi_0 - \varphi_0}{T}\, t\,(1-x) + \sum_{n=1}^{\infty} \frac{1}{4\pi^4 n^4} \frac{\psi_{1n}^{(4)} - \varphi_{1n}^{(4)}}{e^{-\frac{4\pi^2 n^2}{1+4\pi^2 n^2}T} - 1} \left(e^{-\frac{4\pi^2 n^2}{1+4\pi^2 n^2}t} - 1 \right)(1-x)\cos 2\pi n x - \\ &+ \sum_{n=1}^{\infty} \left(\frac{T e^{-\frac{4\pi^2 n^2}{1+4\pi^2 n^2}T}}{\pi^3 n^3 (1+\pi^2 n^2)^2} + \frac{1}{2\pi^5 n^5} \right) \frac{\psi_{1n}^{(4)} - \varphi_{1n}^{(4)}}{e^{-\frac{4\pi^2 n^2}{1+4\pi^2 n^2}T} - 1} \left(e^{-\frac{4\pi^2 n^2}{1+4\pi^2 n^2}t} - 1 \right) \sin 2\pi n x - \\ &+ \sum_{n=1}^{\infty} \frac{1}{4\pi^4 n^4} \frac{\psi_{2n}^{(4)} - \varphi_{2n}^{(4)}}{e^{-\frac{4\pi^2 n^2}{1+4\pi^2 n^2}T} - 1} \left(e^{-\frac{4\pi^2 n^2}{1+4\pi^2 n^2}t} - 1 \right) \sin 2\pi n x + \\ &+ \sum_{n=1}^{\infty} \frac{1}{\pi^3 n^3 \left(1+4\pi^2 n^2\right)^2} \frac{\psi_{1n}^{(4)} - \varphi_{1n}^{(4)}}{e^{-\frac{4\pi^2 n^2}{1+4\pi^2 n^2}T} - 1} t e^{-\frac{4\pi^2 n^2}{1+4\pi^2 n^2}t} \sin 2\pi n x \end{aligned} \tag{30}$$

$$f(x)=2\frac{\psi_0-\varphi_0}{T}(1-x)+\sum_{n=1}^{\infty}4\pi^2n^2\left(\varphi_{1n}-\frac{1}{16\pi^4n^4}\frac{\psi_{1n}^{(4)}-\varphi_{1n}^{(4)}}{e^{-\frac{4\pi^2n^2}{1+4\pi^2n^2}T}-1}\right)(1-x)\cos 2\pi nx+$$
$$+\sum_{n=1}^{\infty}16\pi^2n^2\left(\varphi_{2n}-\left(\frac{Te^{-\frac{4\pi^2n^2}{1+4\pi^2n^2}T}}{4\pi^3n^3(1+\pi^2n^2)^2}+\frac{1}{8\pi^5n^5}\right)\frac{\psi_{1n}^{(4)}-\varphi_{1n}^{(4)}}{e^{-\frac{4\pi^2n^2}{1+4\pi^2n^2}T}-1}\right)\sin 2\pi nx+ \tag{31}$$
$$+\sum_{n=1}^{\infty}\frac{1}{16\pi^2n^2}\frac{\psi_{2n}^{(4)}-\varphi_{2n}^{(4)}}{e^{-\frac{4\pi^2n^2}{1+4\pi^2n^2}T}-1}\sin 2\pi nx-\sum_{n=1}^{\infty}16\pi n\left(\varphi_{1n}-\frac{1}{16\pi^4n^4}\frac{\psi_{1n}^{(4)}-\varphi_{1n}^{(4)}}{e^{-\frac{4\pi^2n^2}{1+4\pi^2n^2}T}-1}\right)\sin 2\pi nx.$$

where, the numbers $\varphi_{1n}^{(4)},\varphi_{2n}^{(4)}$ and $\psi_{1n}^{(4)},\psi_{2n}^{(4)}$ are, respectively, the Fourier coefficients obtained from the Fourier series expansion of the functions $\varphi^{IV}(x)$ and $\psi^{IV}(x)$. According to the condition of Theorem 1, since the functions $\varphi^{IV}(x)$ and $\psi^{IV}(x)$ are continuous on the interval $[0,1]$, and by Bessel's inequality for trigonometric series, the convergence of the following series is deduced:

$$\sum_{n=1}^{\infty}|\varphi_{in}^{(4)}|^2\le C\ \|\varphi^{IV}(x)\|_{L^2(0,1)}^2,\ i=1,\,2, \tag{32}$$
$$\sum_{n=1}^{\infty}|\psi_{in}^{(4)}|^2\le C\ \|\psi^{IV}(x)\|_{L^2(0,1)}^2,\ i=1,\,2. \tag{33}$$

From expressions (32), (33), it follows that the set $\left\{\varphi_{1n}^{(4)},\varphi_{2n}^{(4)},\psi_{1n}^{(4)},\psi_{2n}^{(4)},n=1,\,2,\ldots\right\}$ is bounded. Moreover, the derivatives of the function $u(x,t)$ required in equation (1) are represented, according to series (28), in the following form:

$$u_t(x,t)=2\frac{\psi_0-\varphi_0}{T}(1-x)-\sum_{n=1}^{\infty}\frac{1}{\pi^2n^2(1+4\pi^2n^2)}\frac{\psi_{1n}^{(4)}-\varphi_{1n}^{(4)}}{e^{-\frac{4\pi^2n^2}{1+4\pi^2n^2}T}-1}e^{-\frac{4\pi^2n^2}{1+4\pi^2n^2}t}(1-x)\cos 2\pi nx-$$
$$-\sum_{n=1}^{\infty}\frac{1}{1+4\pi^2n^2}\left(\frac{4Te^{-\frac{4\pi^2n^2}{1+4\pi^2n^2}T}}{\pi n(1+\pi^2n^2)^2}+\frac{2}{\pi^3n^3}\right)\frac{\psi_{1n}^{(4)}-\varphi_{1n}^{(4)}}{e^{-\frac{4\pi^2n^2}{1+4\pi^2n^2}T}-1}e^{-\frac{4\pi^2n^2}{1+4\pi^2n^2}t}\sin 2\pi nx-$$
$$-\sum_{n=1}^{\infty}\frac{1}{\pi^2n^2(1+4\pi^2n^2)}\frac{\psi_{2n}^{(4)}-\varphi_{2n}^{(4)}}{e^{-\frac{4\pi^2n^2}{1+4\pi^2n^2}T}-1}e^{-\frac{4\pi^2n^2}{1+4\pi^2n^2}t}\sin 2\pi nx+ \tag{34}$$
$$+\sum_{n=1}^{\infty}\frac{1}{\pi^3n^3\left(1+4\pi^2n^2\right)^2}\frac{\psi_{1n}^{(4)}-\varphi_{1n}^{(4)}}{e^{-\frac{4\pi^2n^2}{1+4\pi^2n^2}T}-1}e^{-\frac{4\pi^2n^2}{1+4\pi^2n^2}t}\sin 2\pi nx-$$
$$-\sum_{n=1}^{\infty}\frac{4}{\pi n\left(1+4\pi^2n^2\right)^3}\frac{\psi_{1n}^{(4)}-\varphi_{1n}^{(4)}}{e^{-\frac{4\pi^2n^2}{1+4\pi^2n^2}T}-1}te^{-\frac{4\pi^2n^2}{1+4\pi^2n^2}t}\sin 2\pi nx,$$

$$u_{xx}(x,t)=\varphi''(x)-\sum_{n=1}^{\infty}\frac{1}{\pi^2n^2}\frac{\psi_{1n}^{(4)}-\varphi_{1n}^{(4)}}{e^{-\frac{4\pi^2n^2}{1+4\pi^2n^2}T}-1}\left(e^{-\frac{4\pi^2n^2}{1+4\pi^2n^2}t}-1\right)(1-x)\cos 2\pi nx+$$

$$+\sum_{n=1}^{\infty}\frac{1}{\pi^3n^3}\frac{\psi_{1n}^{(4)}-\varphi_{1n}^{(4)}}{e^{-\frac{4\pi^2n^2}{1+4\pi^2n^2}T}-1}\left(e^{-\frac{4\pi^2n^2}{1+4\pi^2n^2}t}-1\right)\sin 2\pi nx-\sum_{n=1}^{\infty}\frac{4}{\pi n(1+4\pi^2n^2)^2}\frac{\psi_{1n}^{(4)}-\varphi_{1n}^{(4)}}{e^{-\frac{4\pi^2n^2}{1+4\pi^2n^2}T}-1}te^{-\frac{4\pi^2n^2}{1+4\pi^2n^2}t}\sin 2\pi nx-$$

$$-\sum_{n=1}^{\infty}\left(\frac{4Te^{-\frac{4\pi^2n^2}{1+4\pi^2n^2}T}}{\pi n(1+\pi^2n^2)^2}+\frac{2}{\pi^3n^3}\right)\frac{\psi_{1n}^{(4)}-\varphi_{1n}^{(4)}}{e^{-\frac{4\pi^2n^2}{1+4\pi^2n^2}T}-1}\left(e^{-\frac{4\pi^2n^2}{1+4\pi^2n^2}t}-1\right)\sin 2\pi nx+$$

$$+\sum_{n=1}^{\infty}\frac{1}{\pi^2n^2}\frac{\psi_{2n}^{(4)}-\varphi_{2n}^{(4)}}{e^{-\frac{4\pi^2n^2}{1+4\pi^2n^2}T}-1}\left(e^{-\frac{4\pi^2n^2}{1+4\pi^2n^2}t}-1\right)\sin 2\pi nx \tag{35}$$

$$u_{xxt}(x,t)=\sum_{n=1}^{\infty}\frac{4}{1+4\pi^2n^2}\frac{\psi_{1n}^{(4)}-\varphi_{1n}^{(4)}}{e^{-\frac{4\pi^2n^2}{1+4\pi^2n^2}T}-1}e^{-\frac{4\pi^2n^2}{1+4\pi^2n^2}t}(1-x)\cos 2\pi nx-$$

$$-\sum_{n=1}^{\infty}\frac{4}{\pi n(1+4\pi^2n^2)}\frac{\psi_{1n}^{(4)}-\varphi_{1n}^{(4)}}{e^{-\frac{4\pi^2n^2}{1+4\pi^2n^2}T}-1}e^{-\frac{4\pi^2n^2}{1+4\pi^2n^2}t}\sin 2\pi nx-$$

$$-\sum_{n=1}^{\infty}\frac{4}{\pi n(1+4\pi^2n^2)^2}\frac{\psi_{1n}^{(4)}-\varphi_{1n}^{(4)}}{e^{-\frac{4\pi^2n^2}{1+4\pi^2n^2}T}-1}e^{-\frac{4\pi^2n^2}{1+4\pi^2n^2}t}\sin 2\pi nx-$$

$$+\sum_{n=1}^{\infty}\frac{16\pi n}{(1+4\pi^2n^2)^3}\frac{\psi_{1n}^{(4)}-\varphi_{1n}^{(4)}}{e^{-\frac{4\pi^2n^2}{1+4\pi^2n^2}T}-1}t\,e^{-\frac{4\pi^2n^2}{1+4\pi^2n^2}t}\sin 2\pi nx+$$

$$+\sum_{n=1}^{\infty}\frac{1}{1+4\pi^2n^2}\left(\frac{4\pi nTe^{-\frac{4\pi^2n^2}{1+4\pi^2n^2}T}}{\left(1+\pi^2n^2\right)^2}+\frac{2}{\pi n}\right)\frac{\psi_{1n}^{(4)}-\varphi_{1n}^{(4)}}{e^{-\frac{4\pi^2n^2}{1+4\pi^2n^2}T}-1}e^{-\frac{4\pi^2n^2}{1+4\pi^2n^2}T}\sin 2\pi nx+$$

$$+\sum_{n=1}^{\infty}\frac{1}{1+4\pi^2n^2}\frac{\psi_{2n}^{(4)}-\varphi_{2n}^{(4)}}{e^{-\frac{4\pi^2n^2}{1+4\pi^2n^2}T}-1}e^{-\frac{4\pi^2n^2}{1+4\pi^2n^2}t}\sin 2\pi nx \tag{36}$$

Now, let’s prove the uniform convergence of the series (30)–(36). Estimating them term by term in absolute value, we derive the following majorant series:

$$|u(x,t)|,\ |u_t(x,t)|\le$$

$$M_1(T)\left(\sum_{n=1}^{\infty}\frac{|\varphi_{1n}^{(4)}|+|\psi_{1n}^{(4)}|}{n^7}+\sum_{n=1}^{\infty}\frac{|\varphi_{1n}^{(4)}|+|\psi_{1n}^{(4)}|}{n^5}+\sum_{n=1}^{\infty}\frac{|\varphi_{1n}^{(4)}|+|\psi_{1n}^{(4)}|+|\varphi_{2n}^{(4)}|+|\psi_{2n}^{(4)}|}{n^4}\right), \tag{37}$$

$$|f(x)|, |u_{xx}(x,t)|, |u_{xxt}(x,t)| \le$$
$$M_2(T)\left(\sum_{n=1}^{\infty}\frac{|\varphi_{1n}^{(4)}|+|\psi_{1n}^{(4)}|}{n^5}+\sum_{n=1}^{\infty}\frac{|\varphi_{1n}^{(4)}|+|\psi_{1n}^{(4)}|}{n^3}+\sum_{n=1}^{\infty}\frac{|\varphi_{1n}^{(4)}|+|\psi_{1n}^{(4)}|+|\varphi_{2n}^{(4)}|+|\psi_{2n}^{(4)}|}{n^2}\right) \quad (38)$$

where $M_i(T),\ i=1,2$ are positive constants. Since the numerical series (37) and (38) converge by comparison, together with (32) and (33), the uniform convergence of the functional series (30)-(36) on the closed domain $\overline{Q}_T$ follows from Weierstrass's theorem. Therefore, the functions $u(x,t), u_t(x,t), f(x), u_{xx}(x,t), u_{xxt}(x,t)$, being constructed from uniformly convergent series of continuous functions, are continuous on $\overline{Q}_T$. The proof of Theorem 1 is complete.

Uniqueness of the Solution

Theorem 2. Suppose that the conditions of Theorem 1 on the existence of a solution to the inverse problem (1)−(4) hold. Then the solution to the inverse problem (1)−(4) is unique.

Proof. Assume, by contradiction, that $(u^1(x,t), f^1(x))$ and $(u^2(x,t), f^2(x))$ are two solutions of the inverse problem (1)−(4). Let $V(x,t)=u^1(x,t)-u^2(x,t)$ and $F(x)=f^1(x)-f^2(x)$, then the functions $V(x,t)$ and $F(x)$ satisfy the following problem

$$\begin{cases} V_t(x,t)-V_{xx}(x,t)-V_{xxt}(x,t)=F(x),\ (x,t)\in Q_T \\ V(x,0)=0, \quad x\in[0,1] \\ V(1,t)=0, V_x(0,t)=V_x(1,t), \quad t\in[0,T] \\ V(x,T)=0, \quad x\in[0,1] \end{cases} \quad (39)$$

The solution of the initial–boundary value problem (39) can be written, using the system of functions (10), in the form of the series

$$V(x,t)=V_0(t)+\sum_{n=1}^{\infty}V_{1n}(t)cos2\pi nx+\sum_{n=1}^{\infty}V_{2n}(t)xsin2\pi nx, \quad (40)$$

$$F(x)=F_0+\sum_{n=1}^{\infty}F_{1n}cos2\pi nx++\sum_{n=1}^{\infty}F_{2n}xsin2\pi nx. \quad (41)$$

where $V_0(t), V_{1n}(t), V_{2n}(t)$ and F_0, F_{1n}, F_{2n} are the Fourier coefficients of $V(x,t)$ and $F(x)$, respectively, with respect to the basis system (10). They satisfy the following boundary value problems for ordinary differential equations:

$$\begin{cases} V_0'(t)=F_0, \\ V_0(0)=0; \quad V_0(T)=0. \end{cases} \quad (42)$$

$$\begin{cases} V_{2n}'(t)+\dfrac{4\pi^2n^2}{1+4\pi^2n^2}V_{2n}(t)=\dfrac{F_{2n}}{1+4\pi^2n^2}, \\ V_{2n}(0)=0; \quad V_{2n}(T)=0. \end{cases} \quad (43)$$

$$\begin{cases} V_{1n}'(t)+\dfrac{4\pi^2n^2}{1+4\pi^2n^2}V_{1n}(t)=\dfrac{F_{1n}}{1+4\pi^2n^2}+\dfrac{4\pi n}{1+4\pi^2n^2}V_{2n}(t)+\dfrac{4\pi n}{1+4\pi^2n^2}V_{2n}'(t), \\ V_{1n}(0)=0; \quad V_{1n}(T)=0. \end{cases} \quad (44)$$

It is easy to see that the solutions of problems (42), (43) and (44) are $V_0(T)=0$, $V_0(t)=0,\ V_{1n}(t)=0,\ V_{2n}(t)=0$ and $F_0=0$, $F_{1n}=0,\ F_{2n}=0$. Hence, $V(x,t)=0$ and $F(x)=0$. Therefore, $u^1(x,t)=u^2(x,t)$ and $f^1(x)=f^2(x)$. Thus, the solution of the inverse problem (1)–(4) is unique. This completes the proof of Theorem 2.

Example 1 (Stationary example)**.** Consider the functions

$$u(x,t)=\sin 2\pi x,\quad f(x)=4\pi^2\sin 2\pi x,$$
$$\varphi(x)=\sin 2\pi x,\quad \psi(x)=\sin 2\pi x$$

This example satisfies all equalities and constitutes an explicit solution of the problem (1) – (8).

Example 2 (Non-stationary example)**.** Let's consider the functions

$$u(x,t)=$$
$$=\left(\frac{C}{(2\pi)^2}+\left(A_0-\frac{C}{(2\pi)^2}\right)e^{-\frac{(2\pi)^2}{1+(2\pi)^2}t}\right)\sin 2\pi x,$$
$$f(x)=C\sin 2\pi x,\qquad \varphi(x)=A_0\sin 2\pi x,$$
$$\psi(x)=A(T)\sin 2\pi x.$$

where $C=const$, $A_0:=A(0)$ and $A(t)=\frac{C}{(2\pi)^2}+\left(A_0-\frac{C}{(2\pi)^2}\right)e^{-\frac{(2\pi)^2}{1+(2\pi)^2}t}$.

Both examples confirm that the conditions of Theorem 1 are satisfied and provide explicit solutions of the inverse problem.

Noting that the investigations in this work are of a theoretical nature, the authors obtained two theorems as the main results. The first theorem establishes the existence of a solution to the inverse problem for a linear pseudoparabolic equation, while the second theorem proves the uniqueness of its classical solution. The obtained results were discussed and approved among young researchers and scientific staff of the Laboratory of Differential Equations and Control Theory of the Faculty of Mechanics and Mathematics at Al-Farabi Kazakh National University.

Discussion

The main object of study in this article is the inverse problem posed for a linear pseudoparabolic equation. Two principal difficulties can be highlighted in this problem: first, the imposition of nonlocal boundary conditions, and second, the determination of the right-hand side depending on the spatial variable. In the case of nonlinear equations, the determination of the right-hand side depending on the spatial variable, or of its coefficient, becomes even more challenging. In fact, research in this direction is almost nonexistent. Therefore, the conclusions obtained in this article regarding the determination of the right-hand side depending on the spatial variable represent valuable results. Consequently, the findings presented in this work will be useful for further studies on numerical solutions and practical applications.

Conclusion

This work considered the inverse problem for a linear pseudoparabolic equation with an additional term representing the effect of external forces depending only on the spatial variable, together with the determination of temperature. The study presented the definition of a classical solution. By means of biorthogonal and orthogonal systems, theorems on the existence, uniqueness and stability of the classical solution to the inverse problem were formulated and rigorously proven in a mathematically precise and clear manner.

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