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Quasi-2D vortex structures in turbulent flows: a Lagranjian model with fractal effects

Abstract. This paper presents a physically motivated model of quasi-two-dimensional vortex structures in turbulent flows. The theory of quasi-two-dimensional turbulence explains many phenomena in geophysical hydrodynamics, since due to the rapid rotation of the Earth, large-scale movements of the atmosphere and ocean almost two-dimensional. Quasi-2D turbulence is approximately two-dimensional and is described by equations containing additional terms. Such additions allow us to take into account weak three-dimensional effects that arise in real conditions, for example, in the atmosphere or ocean. We consider the basic equations for the velocity and pressure fields using the Lagrangian frame and incorporating centrifugal and Coriolis forces, as well as fractal disturbances on the vortex surface. Numerical simulations implemented in MatLab reproduce classical vortex behavior and reveal the influence of fractal corrections on field asymmetry. The model aligns well with existing experimental data and offers a foundation for analyzing energy transport and vortex interactions in stratified or thin-layered turbulent systems.
Keywords: quasi-two-dimensional turbulence, fractal boundary, Lagrangian frame, streamfunction, vortex elements, numerical simulations.

Introduction

Turbulent flows in natural and technical systems often represent coherent vortex structures that display reduced dimensionality due to geometrical constraints or strong rotation. In particular, quasi-two-dimensional turbulence (quasi-2D turbulence) is characteristic of systems where motion is predominantly planar due to strong rotation or confinement [1-3]. Such flows arise in geophysical contexts (the formation of mesovortices or cyclones in the atmosphere and ocean), plasma environments, and thin-layered fluids [4-5].

The most famous result of the theory of 2D turbulence is conclusion that the energy cascade is directed towards large scales and not towards small ones (as in the 3D turbulence case) [6]. Small-scale vortices merge into large, coherent structures in such cases, which play a dominant role in the transport of momentum, heat, and other physical quantities.

In contrast to ideal 2D turbulence governed by strictly two-dimensional Navier-Stokes equations, real quasi-2D systems are influenced by vertical fluctuations, stratification, and surface instabilities [1]. This distinction is particularly important in geophysical and plasma systems, where the emergence of coherent vortex structures is associated with stratification and anisotropy [5, 7]. Some studies have highlighted the role of coherent in mesoscale dynamics [7] and the importance of anisotropic dissipation [2, 8]. Accurately capturing the geometry and dynamics of such systems requires models that account for deviations from perfect symmetry and homogeneity. Understanding the dynamics of such vortices, especially their geometry and interactions, remains an open problem of both theoretical and experimental interest.

Modern approaches to modeling quasi-2D vortices include both numerical techniques and simplified analytical frameworks in which the vortex is treated as the fundamental structural unit of the flow. Such methods are commonly employed in the study of vortex crystal, ring vortices in thin fluid layers, and plasma turbulence [9-14]. Existing research has demonstrated the effectiveness of physically based models in the study of turbulence, using methods from dynamical systems theory, nonlinear physics, and synergetics [15-17].

Recent studies have introduced reduced-order vortex models, spectral closures, and Lattice-Boltzmann-based approaches to model quasi-2D turbulence. However, there remains a gap in accounting for microscale irregularities, such as surface roughness and localized pulsations, that may influence the internal structure of vortices. Several works [18, 19] have shown that introducing fractal corrections can significantly alter energy transport, intermittency, and coherence in turbulent flows. Moreover, earlier measurements have shown that turbulent interfaces, isoscalar surfaces, and dissipative structures exhibit measurable fractal geometry, supporting the validity of fractal descriptions of turbulence [20].

While fractal models are commonly employed in the study of chaotic dynamical systems, their application to the microstructure of turbulent vortices remains relatively unexplored. This work addresses the formation and evolution of a quasi-2D vortex bundle in a submerged viscous medium, using a Largangian framework that incorporates inertial forces and fractal perturbations. The core hypothesis is that surface irregularities of a vortex filament, modeled as scale dependent distortions, can affect the pressure and velocity field structure. Numerical modeling is carried in the MatLab to evaluate the obtained field distributions and particle trajectories. This approach is relevant from a fundamental perspective, contributing to the understanding of self-organization in turbulence, and for practical applications. The model also serves as a basis for constructing simplified turbulence closures in cases where full 3D modeling is computationally inefficient. The proposed model can also be applied to several practical problems. In particular, it may help to describe mesoscale atmospheric vortices, cyclone dynamics, and mixing processes in thin ocean layers.

Theoretical framework

At small scales, the turbulent flow exhibits intermittent vortex filaments. These structures, when spatially localized, demonstrate quasi-2D dynamics dominated by rotation in the plane of the filament cross-section. The predominance of rotation in one direction is due to the stretching of vortex tubes until the rupture, due to tendency of fixed particles of the liquid to move away from each other, i.e., weakening the correlations of the dynamics characteristics. This is one of the main mechanisms for generating turbulence [21]. Less intense, of the order of pulsation, motion along vortex tubes must be taken account in quasi-2D motion.

To describe the structure of quasi-2D turbulent flows, we consider a Lagrangian approach based on the vorticity-stream function formulation. In this setting, the motion of vortex elements is governed by a reduced Navier-Stokes system that incorporates Coriolis effects, pressure gradients, and surface geometry. This formulation is applicable to geophysical and environmental flows where rotation, stratification, and confinement to thin layers lead to quasi-two-dimensional behavior.

Figure 1 schematically explains what has been said. In cross-section 2, where the core radius of the vortex satisfies $r_{02} < r_{01}$, the angular velocity Ω_{02} is greater than Ω_{01} ($\Omega_{02} > \Omega_{01}$), in accordance with the law of conservation of angular momentum. Quasi-2D motion implies a coupling between rotation in the xy-plane and pulsation motion induced by the nonstationary surface perturbation $\eta = \eta(x, y, t)$ of any cross-section (Fig.1).

Such behavior can be simulated using layered ring elements with a distributed angular velocity subject to inertial forces (centrifugal and Coriolis) and internal structural restrictions. The model includes $\eta(\theta)$ fluctuating superficial distrurbances interpreted as fractal corrections. The inertial forces per unit mass are given by centrifugal $\vec{F}_c$ and Coriolis $\vec{F}_{cor}$:

$$\vec{F}_c = \vec{r}_0 \Omega_0^2, \quad \vec{F}_{cor} = -2\left[\vec{\Omega}_0 \vec{v}\right]. \tag{1}$$

These forces are expressed through the circulation of the vortex nucleus Γ_0:

$$\Gamma_0 = \oint v_0 dr_0 = \oint \Omega_0 r_0 dr = \Omega \int_0^{4\pi} r_0 dr_0 = 2\pi \Omega_0 r_0^2. \tag{2}$$

Γ_0 is condition for stationary generation of turbulence.

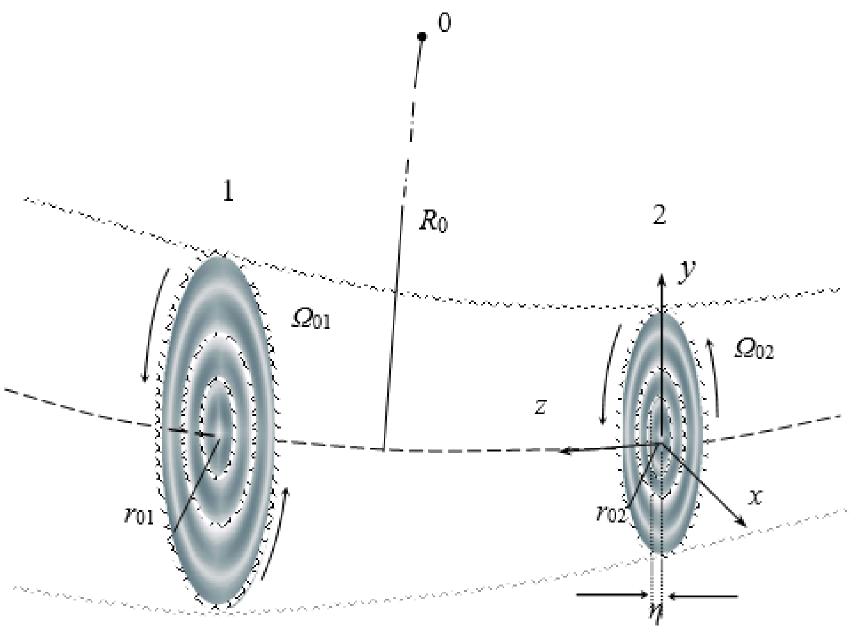


Figure 1 – Scheme of the quasi-2D vortex filament

R_0 is radius of the vortex filament, r_{01} and r_{02} are core radii of the vortex in different cross-sections of the filament, Ω_{01} and Ω_{02} are angular velocities in cross-sections 1 and 2, respectively, η is perturbation of the vortex surface level.

Using Kelvin's circulation theorem [22] and accounting for viscosity via modified pressure-density relations, we derive the governing equations:

$$\frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \Omega^2 r + 2\Omega v, \quad (3)$$

$$\frac{dv_r}{dr} + \frac{v_r}{r} + \frac{dv_z}{dz} = \eta(r). \quad (4)$$

Eq.(3) is momentum balance and true in the approach of a vortex ring in cylindrical symmetry. Eq.(4) is continuity and represents mass conservation in cylindrical coordinates. Here, the additional term $\eta(r)$ accounts for small-scale fractal perturbations to the vortex cross-sections, effectively modifying the divergence of the velocity field and modeling sourface roughness or pulsational instability of the vortex interface. This approximation allows for a better fit of the model with experimentally observed vortex structures in which spiral or wave-like distortions are observed.

The distribution of velocity v_r and pressure associated with $\eta(r)$ in a turbulent vortex can be determined using these equations. A vortex with velocity distribution is chosen as a reference system:

$$v(r) = v_0 \cdot \frac{r_0}{r}, \qquad r \geq r_0. \quad (5)$$

For $r \leq r_0$, the velocity distribution is the same as when the liquid rotates as a solid body around a selected axis with an angular velocity, i.e. the same velocity distribution as a for the Rankine vortex.

The pressure in the vortex associated with centrifugal forces is defined as:

$$p = \frac{\rho r_0 \Omega_0^2}{F_0} = p_0 + \rho r_0 \Omega_0^2 \eta \quad (6)$$

where ρ is density of the liquid. It is accepted here that fractal measures: volume and area can be represented as

$$V(\eta) = V_0 + F_0\,\eta, \quad F(\eta) = F_0 + r_{0*}\,\eta, \quad (7)$$

where V_0, F_0 and $r_{0*} = \pi r_0$ are some effective geometric characteristics of the vortex.

Numerical implementation

To solve the model equations numerically, we consider a qusi-2D vortex in the Rankine vortex approximation. The angular velocity is defined as linear inside the core ($r \leq r_0$)

$$v_\theta = \frac{\Gamma}{2\pi r_0^2} r,$$

and inversely proportional outside ($r > r_0$)

$$v_\theta = \frac{\Gamma}{2\pi r}.$$

The corresponding pressure gradient is computed from centrifugal balance and integrated numerically. Fractal perturbations are introduced by modulating the core radius $r_0(\theta) = r_0 + noise$, where $noise$ is random noise. Using MatLab, we implemented: a) 1D plots of $v_\theta(r)$ and $p(r)$; b) 2D visualizations (quiver and contour plots) for velocity and pressure fields; and c) Lagrangian particle tracking to simulate advection a in the vortex.

To visualize the effect of fractal distortions, we begin by constructing two surface geometries: (a) a smooth cylindrical surface with constant radius r_0 (it is Kolmogorov-type vortex), and (b) a deformed surface with a radius $r_0(\theta) = r_0 + noise$, where $noise$ is white noise smoothed by convolution:

$$\eta(z) = \sum_{n=1}^{N} a_n \sin(k_n z + \phi_n), \quad k_n \sim 2^n, a_n \sim \frac{1}{2^n}.$$

The radial displacement $\eta(z)$ is modeled as a sum of multiscale oscillations, resulting in a rough, non-periodic vortex boundary.

Results and Discussion

Geometries of classical and fractal-like surfaces are shown in Figure 2. This illustration emphasizes the potential impact of small-scale deformations on the structure of a quasi 2D vortex.

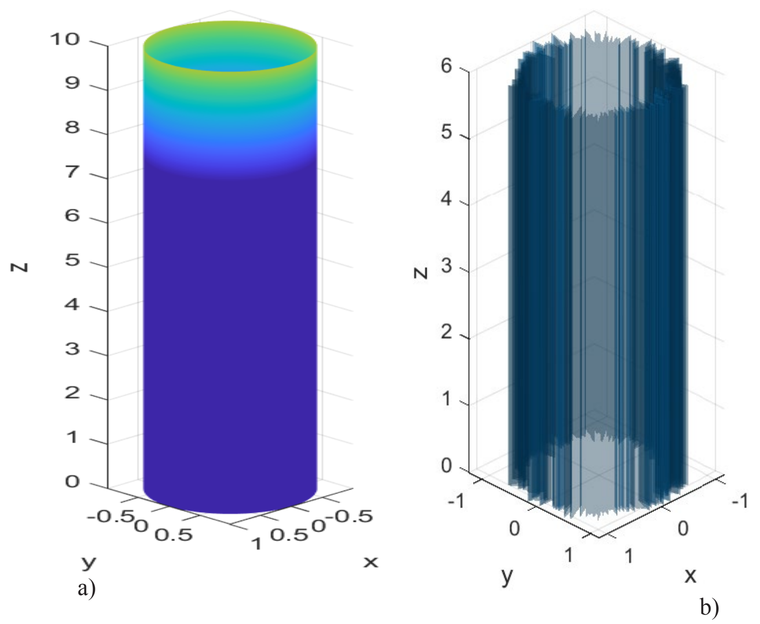
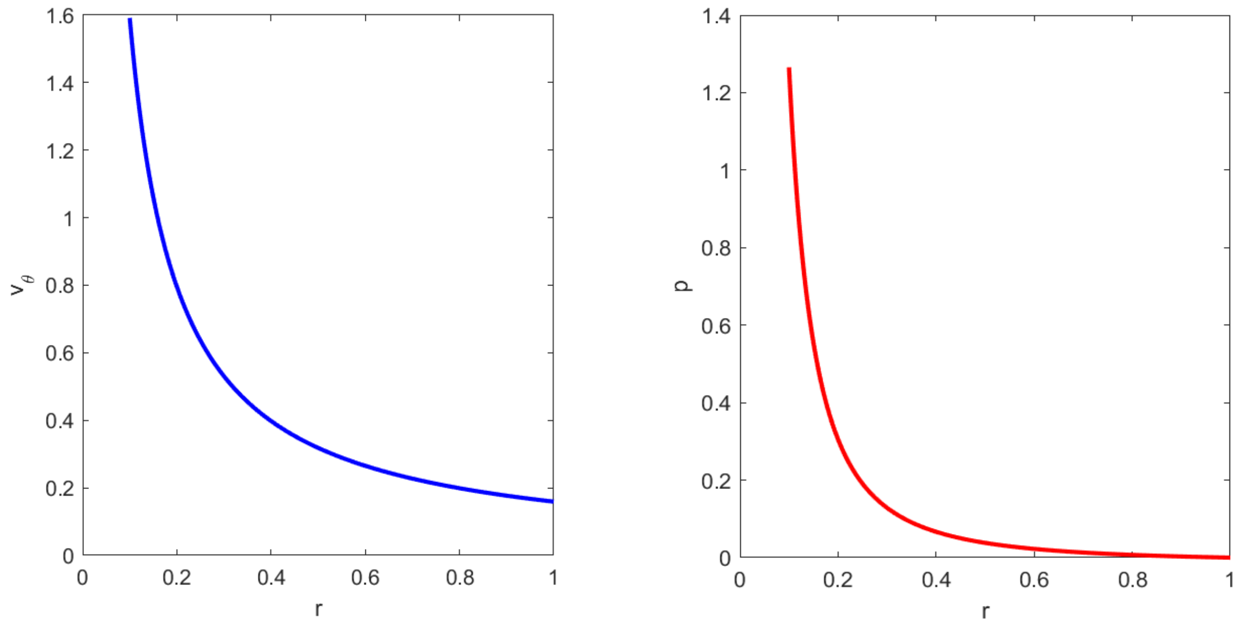


Figure 2 – Comparison between a smooth and fractally deformed vortex surfaces. Fractal deformation induces surface complexity that breaks axial symmetry of the vortex boundary. a) classical cylindrical surface with constant radius r_0, and b) vortex with a fractal deformed surface

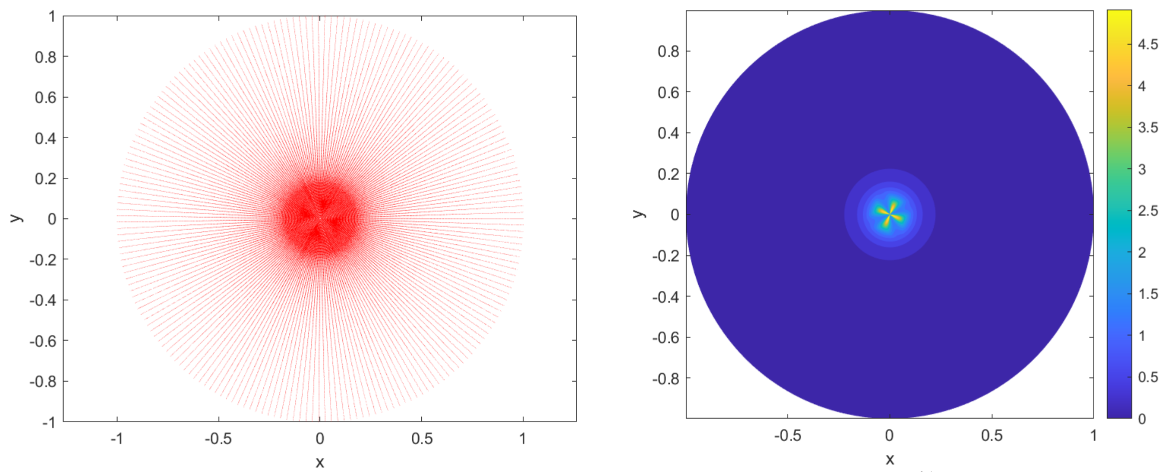
Velocity and pressure fields for the Rankine vortex model were computed using MatLab, with results shown in Figure 3. As expected, the angular velocity distribution (Fig. 3a) is linear inside the core (for $r < r_0 = 0.1$), corresponding to solid-body rotation, and decays as $1/r$ outside the core. This profile is essential for identifying the rotation regime and estimating the effective radius of self-organization. The pressure field (Fig. 3b) decreases toward the center, with a maximum at the periphery, consistent with experimental observations of ring vortices reported by [23, 24].



a) b)

Figure 3 – Profiles of angular velocity (a) and pressure (b) in the Rankine vortex model

Next, we introduce fractal deformations to the vortex surface and perform 2D visualization of the velocity and pressure fields. Figure 4a shows the quiver plot of the velocity field, while Figure 4b presents the pressure distribution using a contour plot. The domain is discretized in polar coordinates, with the effective core radius computed for each angular direction, followed by mapping to Cartesian space for visualization. MatLab's built-in functions quiver and contour were used for these visualizations.



a) b)

Figure 4 – Velocity (a) and pressure (b) field under fractal perturbations.
Asymmetry is visible near the core

The addition of fractal distortions introduces mild asymmetries, especially near the vortex core. These effects are similar to patterns observed in laboratory experiments where vortex boundaries exhibit undulating or irregular shapes when visualized with tracer particles [2]. This supports the application of fractal modeling to describe microstructural features of coherent vortices.

To demonstrate advective transport, we tracked Lagrangian trajectories of particles in the quasi-2D vortex field with fractal perturbations. Figure 5 illustrates the evolution of particles (red dots) initially placed on a regular grid within the central domain. The particles are gradually drawn into curved path near the center, forming closed trajectories that indicate sustained circulation. This behavior confirms the presence of a coherent vortex shell, a hallmark of quasi-2D vortex dynamics. Similar structures have been documented in thin-layer flow experiments and numerical studies of 2D turbulence [1].

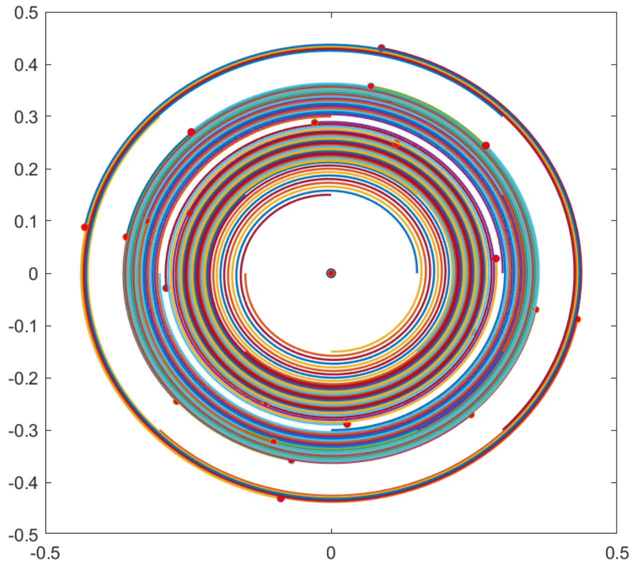


Figure 5 – Lagrangian particle trajectories in a quasi-dimensional vortex with fractal surface: formation of coherent, enclosed paths suggests robust vortex trapping

Overall, the simulation results are consistent with experimental findings and theoretical predictions. In particular, the velocity and pressure profiles agree with experimental data from shallow-layer vortex studies [2]. The flow structure also reproduces key features described in recent studies of 2D and qusi-2D turbulence [1, 10].

The velocity field analysis shows deviations from axial symmetry due to the imposed surface perturbations. The streamfunction relief demonstrates complex topography typical of turbulent cores. Compared to idealized Rankine vortices, the fractal deformed model offers improved representation of surface irregularities and secondary flow effects. Further analysis is warranted to investigate the energy spectra and vorticity transport mechanisms within the distorted boundary layer.

Conclusion

In this study, a simplified Lagrangian model were developed for simulating quasi-two-dimensional vortex structures with fractal surface perturbations. Analytical solutions were derived based on the stream function-vorticity formulation, incorporating rotational and pressure-driven effects. The resulting streamfunction fields revealed localized, ring-like vortex structures exhibiting scale-invariant characteristics when modulated by logarithmically oscillating functions.

Modeling performed using MatLab confirmed that the geometric perturbations at vortex boundaries, designed to mimic natural fractal deformation, significantly affect the internal structure and symmetry of the flow. These effects are particularly important for understanding turbulent mixing and energy transport in confined, rotating, or stratified environments such as planetary atmospheres, oceans, or magnetized plasmas.

The presented approach demonstrates the potential of structural modeling in revealing important features of turbulence beyond traditional spectral methods. Here we focused primarily on the theoretical formulation and numerical verification. In the future work, we plan to conduct a more detailed analysis of the proposed model's potential applications to geophysical and plasma systems, such as atmospheric vortices, ocean currents, and magnetic plasma turbulence. In particular, the model can be applied to studying mesoscale vortex formation in the atmosphere, cyclone dynamics, and transport processes in thin ocean layers. It may be also be useful for analyzing confinement and mixing effects in laboratory plasma devices.

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