International Journal of Mathematics and Physics 6, №1, 30 (2015)

UDC 510.54

Issakhov As.A.

Faculty of Mechanics and Mathematics,
Al-Farabi Kazakh National University, Almaty, Kazakhstan
e-mail: asylissakhov@mail.ru

An example on existence of a $\sum_{n+2}^{0}$-computable family of total functions whose rogers semilattice contains an ideal without minimal elements

Abstract. We study computable families of total functions of any level of the Kleene-Mostowski hierarchy above level 1 and try to find elementary properties of the Rogers semilattices that are different from the properties of the classical Rogers semilattices for families of computable functions. It is known that on first level of the arithmetical hierarchy the Rogers semilattice of any computable family of total functions contains no ideal without minimal elements, [1]. In this article we show an example how to build $\sum_{n+2}^{0}$-computable family of total functions whose Rogers semilattice contains an ideal without minimal elements, $n \in \omega$.

Keywords: $\sum_{n+2}^{0}$-computable numbering, $\sum_{n+2}^{0}$-computable family,computability relative to the oracle $\emptyset^{(n+1)}$, minimal numbering, Rogers semilattice,numerical equivalence, positive equivalence, ideal.

Introduction

We refer the reader to [1, 2, 3] for the standard notions and notations in algorithm theory and in numbering theory.

Let F be a $\sum_{n+2}^{0}$-computable family of total functions, where $n \in \omega$. A numbering $\alpha : \omega \to F$ is called $\sum_{n+2}^{0}$-computable if the binary function $\alpha(n)(x)$ is $\sum_{n+2}^{0}$-computable, which means that the function $\alpha(n)(x)$ is computable relative to the oracle $\emptyset^{(n+1)}$, (see [4, 5]). A family F is called $\sum_{n+2}^{0}$-computable if it has an $\sum_{n+2}^{0}$-computable numbering. The notion of a $\sum_{1}^{0}$-computable numbering coincides with the classical notion of a computable numbering of a family of computably enumerable sets in [1]. Let $Com_{n+2}^{0}(F)$ be the set of all $\sum_{n+2}^{0}$-computable numberings of a family F. If α and β are two numberings of a same family F, then we say that the numbering α is reducible to the numbering β, if there is a computable function f such that $\alpha = \beta f$, and we write this symbolically as $\alpha \le \beta$. If $\alpha \le \beta$ and $\beta \le \alpha$ then the numberings α and β are called equivalent, written as $\alpha \equiv \beta$. Denote by $\deg(\alpha)$ the degree of α, i.e. the set $\{\beta \mid \beta \equiv \alpha\}$ of numberings. The reducibility relation of numberings is a pre-order relation on $Com_{n+2}^{0}(F)$, and it induces a partial order relation on a set of degrees of the numberings in $Com_{n+2}^{0}(F)$, which is usually also denoted by $\le$. The partially ordered set $\Re_{n+2}^{0}(F) =_{def} \left\langle \{\deg(\alpha) \mid \alpha \in Com_{n+2}^{0}(F)\}, \le \right\rangle$ is an upper semilattice and called the Rogers semilattice of the family F, [5].

A numbering α of a family F is called minimal if for any numbering β of F, reducibility of β to α implies that α is reducible to β. The numerical equivalence θ_α of a numbering α is defined as follows: $\theta_\alpha =_{def} \{(x, y) \mid \alpha(x) = \alpha(y)\}$.

An equivalence relation ε is said to be positive if ε is computably enumerable. Denote by $[W]_\varepsilon$

the set of all numbers, which are ε-equivalent to some element from *W*. For the further undefined notions, which are related to relativized computable numberings we refer to [4, 5, 6].

Results

It is well known many infinite families of c.e. sets whose Rogers semilattice contains an ideal without minimal elements, for instance, the family of all c.e. sets, [1]. Moreover, there exists a computable family of c.e. sets whose Rogers semilattice has no minimal elements at all, [7, 8]. In opposite to the case of the families of c.e. sets, for every computable numbering *α* of an infinite family *F* of computable functions, there is a Friedberg numbering of *F* which is reducible to *α*, [1]. This means that the Rogers semilattice of any computable family of total functions from level 1 of the arithmetical hierarchy contains no ideal without minimal elements.

In [7] Badaev proved the criterion for numberings to be minimal:

Theorem 1. ([4]). Let $\alpha:\omega\to S$ be a numbering of an arbitrary set *S*. Then the following statements are equivalent:

a) α is a minimal numbering;

b) for any c.e. set $W\subseteq\omega$ such that $[W]_{\theta_\alpha}=\omega$, there exists a positive equivalence relation $\varepsilon\subseteq\theta_\alpha$ such that $[W]_\varepsilon=\omega$.

And, this result was extended in [9] up to a criterion for a numbering not bounded any minimal numbering:

Theorem 2. ([9]). Let *α* be a numbering of an arbitrary set *S*. Then *S* has a minimal numbering, which is reducible to *α* if and only if there exists a c.e. set *W* such that

a) $\alpha(W) = S$ and

b) for any c.e. set $V\subseteq W$ where $\alpha(V) = S$ there is a positive equivalence relation ε such that ε restricted on *W* is a subset of θ_α and $W\subseteq[V]_\varepsilon$ hold.

Indeed theorem 2 is a corollary of theorem 1, andwe just reformulate theorem 2:

Theorem 3. Let *α* be a numbering of an arbitrary set *S*. Then there is no minimal numbering of *S* that is reducible to *α* if and only if, for every c.e. set *W*, if $\alpha(W) = S$ then there exists a c.e. set *V* such that $\alpha(V) = S$ and, for every positive equivalence ε, either ε restricted on *W* is not a subset of θ_α or $W\not\subseteq[V]_\varepsilon$.

These criteria (theorem 1-3) hold for numberings of any set, not only for numberings of families of total functions. The next theorem is based on theorem 3 and relates to [10]. It is an example of a Σ^0_{n+2}-computable family whose Rogers semilattice contains an ideal without minimal elements. Before formulating theorem 4, we note that every Rogers semilattice of a Σ^0_{n+2}-computable family *F* contains the least element if *F* is finite, [1], and infinitely many minimal elements, otherwise, [5].

Theorem 4. For every $n\in\omega$, there exists a Σ^0_{n+2}-computable family of total functions whose Rogers semilattice contains an ideal without minimal elements.

Proof. By theorem 3 it is clear that it is enough to construct numbering *α* of Σ^0_{n+2}-computable family which satisfy the following condition:

$$\forall W_i\big(\alpha(W_i)=\alpha(\omega)\to\exists V_i\subseteq W_i\big(\alpha(V_i)=\alpha(\omega)\wedge\forall\varepsilon_j(\varepsilon_j\restriction W_i\not\subseteq\theta_\alpha\vee W_i\not\subseteq[V_i]_{\varepsilon_j})\big)\big),\quad(1)$$

where W_i and V_i are c.e. sets, ε_j is a positive equivalence.

Construction of α:

Our Σ^0_{n+2}-computable family will consist of constant functions and functions, which differ from constant functions exactly on one point.

On stage *i*, we ask oracle $\emptyset'$ about belonging of elements *α* and *b* to W_i, where we denote $\alpha = 2 < i,j >$ and $b = 2 < i,j > +1$ for any $j = 0, 1, 2, \ldots$:

If $a\notin W_i\wedge b\notin W_i$ for some j, then we put $\alpha(a)(0)=i+1,\alpha(a)(s+1)=i,$, for $s=0,1,2,\ldots$;

$\alpha(b)(0)=i+1,\alpha(b)(s+1)=i$, for $s=0,1,2,\ldots$;

If $a\in W_i\wedge b\notin W_i$ for some j, then we put $\alpha(a)(s)=i$, for $s=0,1,2,\ldots$;

$\alpha(b)(0)=i+1,\alpha(b)(s+1)=i$, for $s=0,1,2,\ldots$;

If $a\notin W_i\wedge b\in W_i$ for some j, then we put $\alpha(a)(0)=i+1,\alpha(a)(s+1)=i$, for $s=0,1,2,\ldots$;

$\alpha(b)(s) = i$, for $s = 0,1,2,\ldots$;

And finally, if $a \in W_i \wedge b \in W_i$, then we construct $\alpha(a)$ and $\alpha(b)$ step by step:

Step 0, we put $\alpha(a)(0) = \alpha(a)(1) = i$ and $\alpha(b)(0) = i$ (indeed, $\alpha(b)$ will be constructed like $\alpha(a)$, but little bit slowly).

Step $s+1$, on this step we know that $\alpha(a)(0) = \alpha(a)(1) = \ldots = \alpha(a)(s+1) = i$ and we know the values of

$$\alpha(b)(0), \alpha(b)(1), \ldots, \alpha(b)(s). \qquad (2)$$

If in (2) there is the value $i+1$, then we put $\alpha(a)(s+2) = \alpha(b)(s+1) = i$;

If in (2) there is no value $i+1$, then we check the following condition: $(a,b) \in \varepsilon_j^{s+1}$?

if "yes", then $\alpha(a)(s+2) = i$ and $\alpha(b)(s+1) = i+1$,

if "no", then $\alpha(a)(s+2) = \alpha(b)(s+1) = i$.

We put $V_i = W_i \setminus \widehat{V}_i$, where $\widehat{V}_i = \{x \mid x = = 2 < i, j >, j \geq 1 \; or \; x = 2 < i, j > +1, j \geq 0\}$ – computable set, then the condition (1) is holds for numbering α.

Checking:

If for some j (with fixed i) one of a or b doesn't belong to W_i, then $\alpha(W_i) \neq \alpha(\omega)$;

If for any j, a and b belong to W_i, then we have two cases.

1. $(a,b) \in \varepsilon_j$ for some j, then by construction $\alpha(a) \neq \alpha(b)$, i.e. $\varepsilon_j \upharpoonright W_i \not\subseteq \theta_\alpha$.

2. $(a,b) \notin \varepsilon_j$ for any j, then by construction $\alpha(a)(x) = \alpha(b)(x)$ for any $x \in \omega$ and there is only one index-number $\alpha = 2 < i, 0 >$ of function $\alpha(a)(s) = i$ in V_i, what means that the corresponding $b = 2 < i, 0 > +1 \in W_i$, but $b = 2 < i, 0 > +1 \notin [V_i]_{e_j}$, i.e. $W_i \not\subseteq [V_i]_{\varepsilon_j}$.

Conclusion

It is the next step in studying the generalized computable families of total functions and their generalized computable numberings. Constructed example of $\sum_{n+2}^{0}$-computable family of total functions whose Rogers semilattice contains an ideal without minimal elements, where $n \in \omega$, shows that the elementary properties of the corresponding Rogers semilattices are very rich.

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