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Differential-algebraic equations with boundary terms

Abstract. In the referenced study, a differential equation (DE) exhibiting a hybrid structure is examined. The principal objective of this manuscript is to determine the feasibility of substituting the given supplementary boundary conditions with alternative equivalent conditions. This is achieved through the establishment and proof of four theorems, providing a rigorous foundation for the proposed substitutions. Incipiently, the existing (3) conditions are considered in a nonhomogeneous context. Subsequently, new conditions, denoted as (7), are introduced. These newly formulated conditions are demonstrated to be equivalent to the original ones, ensuring the unique solvability of the hybrid-structured system labeled as (1). The system under consideration is characterized as hybrid due to the presence of both unknown $y(x)$ and algebraic components. This dual nature necessitates a nuanced approach to boundary condition formulation and analysis. The methodology employed in this study underscores the importance of flexibility in boundary condition specification, particularly in complex or hybrid configurations. By establishing the equivalence of different boundary conditions, article provides valuable insights into the solvability and analysis of such frameworks. Furthermore, the study meticulously details prior research in this domain, delineating the specific conditions and configurations previously explored. This comprehensive review situates the current manuscript within the broader context of hybrid DE analysis.

Key words: boundary, hybrid, dissipative, DAE, BVP.

Introduction

The term differential-algebraic equation (DAE) was first introduced in the title of a 1971 publication by Gear [1], and that same year marked the release of his seminal monograph [2], wherein he explores examples arising from the analysis of electrical circuits. Two principal domains of application–namely, circuit analysis and mechanical frameworks with conditions–have remained among the primary driving forces behind the advancement of the theory of DAEs. Even today, more than 150 years after Kirchhoff's foundational contributions, circuit analysis continues to serve as a significant impetus for the development of DAEs. In this context, the interplay between modeling and mathematical analysis is of particular importance. Readers seeking a more comprehensive understanding are referred to the foundational literature [3]. Between 1989 and 1996, the field of DAE theory and its numerical treatment experienced a period of rapid growth, during which numerous research groups in both mathematics and engineering began to engage with this emerging area of inquiry. Those interested in delving deeper into the subject and the extensive body of manuscript accumulated over the years may consult the monographs [4]–[8] as well as the review article [9]. During the formative years of DAE research, regularization emerged as a widely employed technique for transforming the algebraic components into DEs. Motivated by physical scenarios such as stiff springs or parasitic phenomena in electrical circuits, a number of researchers pursued investigations in this direction. Another fruitful avenue of study involves commencing with a singularly perturbed ordinary DE, discretizing it, and subsequently analyzing the asymptotic behavior of its exact solutions in the limiting case. In 2006, Mehrmann published a volume [10] that offered fresh insights into several pertinent topics, including boundary value problems (BVP) for differential-algebraic equations.

The domain of DEs subject to boundary conditions–commonly referred to as boundary-DEs–

has been extensively surveyed from the early 20th century to the present day, with particular emphasis on recent developments. Among the earliest contributors in this field was R. Phillips, who engaged directly with boundary-differential operators. His investigations provided a rigorous conceptual framework for understanding the structural nature of such equations. Specifically, he explored methodologies involving minimal and maximal operators, as well as their dissipative extensions–an approach that is intrinsically linked to the formulation of boundary DEs. In his seminal scrutiny [11], examined maximally dissipative operators, thereby underscoring the necessity of extending purely differential operators to encompass boundary-differential formulations. This realization highlights the fact that, in numerous cases, differential operators naturally evolve into boundary-differential operators–i.e., operators that incorporate both differential components and boundary-defined terms. Further contributions, such as those in [12], consider formulations that involve not only differential expressions but also boundary functionals that explicitly depend on the boundary values of the solution. Boundary value problems form an important area of applied mathematics for explicitly given ordinary differential equations; see, for example, [13-16]. The study of the properties of singular perturbations of some differential operators and well-defined restrictions is devoted to the works [17, 18]. This assertion is even more applicable in the context of differential-algebraic equations (DAEs). The studies [19-25] predominantly focus on BVPs involving conditions prescribed at two distinct boundary spots. These investigations play a pivotal role in advancing the theoretical understanding and computational treatment of DAEs under boundary conditions.

The present study is devoted to the analysis of boundary DEs incorporating algebraic components on a finite interval, representing a hybrid structure that combines DAEs with boundary terms. The first part provides a comprehensive classification of all possible linear, well-posed problems within this framework. In the second part, inverse problems corresponding to the aforementioned class of linear, well-posed boundary-DEs with algebraic components on a finite domain are formulated and examined.

Let us consider a regimen composed of a boundary DE involving algebraic components on a finite interval with respect to the unknown function $y(x)$, together with an associated regimen of linear algebraic equations for the scalar parameters $\mu_1,\dots,\mu_s$.

$$\begin{cases} l(y,\vec{\mu}) \equiv \dfrac{d}{dx}\left(\dfrac{dy(x)}{dx} + \sum_{i=1}^{k} h_i(x)U_i(y) + \sum_{j=1}^{s}\mu_j q_j(x)\right) + r_1(x)\dfrac{dy(x)}{dx} + r_0(x)y(x) = f(x),\ 0<x<1, \\ \theta_m(y,\vec{\mu}) \equiv \sum_{j=1}^{s} a_{mj}\mu_j + b_m y(1) + c_m y(0) + d_m\dfrac{dy(1)}{dx} + e_m\dfrac{dy(0)}{dx} = \omega_m,\ m=1,\dots,s, \end{cases} \tag{1}$$

whereby, $r_k \in C^{(k)}[0,1], k=0,1$.

Let $\{h_1(\cdot),\dots,h_k(\cdot)\}$ and $\{q_1(\cdot),\dots,q_s(\cdot)\}$ denote prescribed collections of functions. Particular attention must be given to the specification of the boundary functionals $U_1(\cdot),\dots,U_k(\cdot)$. In the case where the functions $\{h_1(\cdot),\dots,h_k(\cdot);q_1(\cdot),\dots,q_s(\cdot)\}$ exhibit smoothness in neighborhoods of the $x=0$ and $x=1$, the $U_1(\cdot),\dots,U_k(\cdot)$ may be expressed in the form

$$U_i(\mathrm{y}) = \sum_{j=1}^{2}\left(\alpha_{ij}y^{(j-1)}(0) + \beta_{ij}y^{(j-1)}(1)\right), i=1,\dots,k, \tag{2}$$

where α_{ij}, β_{ij} are complex coefficients.

We now augment the regimen of equations (1) by incorporating with an augmented set of admissibility conditions:

$$\begin{cases} V_1(y,\vec{\mu}) = \alpha_1 y(1) + \beta_1 y(0) + \gamma_1 \dfrac{dy(1)}{dx} + \delta_1 \dfrac{dy(0)}{dx} = 0, \\ V_2(y,\vec{\mu}) = \alpha_2 y(1) + \beta_2 y(0) + \gamma_2 \dfrac{dy(1)}{dx} + \delta_2 \dfrac{dy(0)}{dx} = 0, \end{cases} \tag{3}$$

Within the present analytical framework, the coefficient matrix associated with the end spot operator

$$\Gamma = \begin{bmatrix} \alpha_1 & \beta_1 \ \gamma_1 & \delta_1 \\ \alpha_2 & \beta_2 \ \gamma_2 & \delta_2 \end{bmatrix}$$

is assumed to attain full rank equal to two, thereby ensuring the non-degeneracy of the corresponding endpoint operator and the linear independence of the associated endpoint conditions in the admissible specification space. The principal objective addressed in the present manuscript may be formulated as follows:

Are there any additional specifications, apart from those specified in (3), that ensure the regimen of equations (1) with these supplementary specifications admits a uniquey solution for all $f(x) \in L_2(0,1), \omega_1 \in \mathbb{C}, \dots, \omega_s \in \mathbb{C}$? Provide the general form of such an augmented set of admissibility conditions.

The problem at hand, in the context of DEs, is well-established in the [26].

In this section, we will introduce our salient notables.

Theorem 1. Assume that $\Delta = \det \begin{pmatrix} A \\ C_0 \end{pmatrix} \neq 0$.

Under this non-degeneracy condition the (1), (3) is uniquely solvable for all right-hand sides $f(x) \in L_2(0,1), \omega_1 \in \mathbb{C}, \dots, \omega_s \in \mathbb{C}$. Then, the solution to the problem (1), (3) is given by the formula:

$$\begin{cases} y_0(x) = \sum_{m=1}^{s} \omega_m \left(\dfrac{H_{m1}}{\Delta} y_1(x) + \dfrac{H_{m2}}{\Delta} y_2(x) + \sum_{i=1}^{k} \dfrac{H_{(s+i)m}}{\Delta} \widetilde{H_i}(x) + \sum_{j=1}^{s} \dfrac{H_{jm}}{\Delta} \widetilde{Q_j}(x) \right) + \\ \int_0^x \dfrac{\begin{vmatrix} y_1(t) & y_2(t) \\ y_1(x) & y_2(x) \end{vmatrix}}{\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}} f(t)dt + \int_0^1 \dfrac{\begin{vmatrix} y_1(t) & y_2(t) \\ y_1(1) & y_2(1) \end{vmatrix} + \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(1) & y_2'(1) \end{vmatrix}}{\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}} f(t)dt \left(\dfrac{A_1}{\Delta} y_1(x) + \right. \\ \left. \dfrac{A_2}{\Delta} y_2(x) + \dfrac{A_3}{\Delta} \sum_{i=1}^{k} \widetilde{H_i}(x) + \dfrac{A_4}{\Delta} \sum_{j=1}^{s} \widetilde{Q_j}(x) \right), \\ \mu_j^0 = \dfrac{1}{\Delta} \left(\sum_{m=1}^{s} \omega_m H_{jm} + A_4 \int_0^1 \dfrac{\begin{vmatrix} y_1(t) & y_2(t) \\ y_1(1) & y_2(1) \end{vmatrix} + \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(1) & y_2'(1) \end{vmatrix}}{\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}} f(t)dt \right), j = 1, \dots, s. \end{cases} \tag{4}$$

The entities denoted by $H_{m1}, H_{m2}, H_{(s+i)m}$, H_{jm}; A_1, A_2, A_3, A_4 constitute a collection of determinant expressions, the explicit constructions of which are expounded in Appendix 1.

$$\begin{bmatrix} A \\ C_0 \end{bmatrix} = \begin{bmatrix} a_{11} - V_1(\widetilde{Q_1}) & \dots & a_{1s} - V_1(\widetilde{Q_s}) & -V_1(\widetilde{H_1}) & \dots & -V_1(\widetilde{H_k}) & V_1(y_1) & V_1(y_2) \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ a_{s1} - V_1(\widetilde{Q_1}) & \dots & -a_{ss} - V_1(\widetilde{Q_s}) & -V_1(\widetilde{H_1}) & \dots & -V_1(\widetilde{H_k}) & V_1(y_1) & V_1(y_2) \\ -U_1(\widetilde{Q_1}) & \dots & -U_1(\widetilde{Q_s}) & -U_1(\widetilde{H_1}) - 1 & \dots & -U_1(\widetilde{H_k}) & U_1(y_1) & U_1(y_2) \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ -U_k(\widetilde{Q_1}) & \dots & -U_k(\widetilde{Q_s}) & -U_k(\widetilde{H_k}) - 1 & \dots & -U_k(\widetilde{H_k}) - 1 & U_k(y_1) & U_k(y_2) \\ -V_2(\widetilde{Q_1}) & \cdots & -V_2(\widetilde{Q_s}) & -V_2(\widetilde{H_1}) & \cdots & -V_2(\widetilde{H_k}) & V_2(y_1) & V_2(y_2) \\ -V_3(\widetilde{Q_1}) & \cdots & -V_3(\widetilde{Q_s}) & -V_3(\widetilde{H_1}) & \cdots & -V_3(\widetilde{H_k}) & V_3(y_1) & V_3(y_2) \end{bmatrix}.$$

It is pertinent to observe that the matrix A is induced by the DE, whereas the matrix C_0 is a consequence of the boundary conditions. The constituent entries of this $\begin{bmatrix} A \\ C_0 \end{bmatrix}$ are meticulously expounded in Appendix 2.

Within the framework of the problem delineated by (1) and (3), the boundary conditions articulated in (3) are characterized by homogeneity. Let us examine the scenario in which the supplementary conditions (3) are non-homogeneous as well.

Theorem 2. Provided that $\Delta = \det\begin{pmatrix} A \\ C_0 \end{pmatrix} \neq 0$, then for any vectors $\vec{p}$ and for all $f(x) \in L_2(0,1), \omega_1 \in \mathbb{C}, \dots, \omega_s \in \mathbb{C}$, the (1) with non-homogeneous supplementary conditions

$$\begin{cases} \alpha_1 y(1) + \beta_1 y(0) + \gamma_1 \dfrac{dy(1)}{dx} + \delta_1 \dfrac{dy(0)}{dx} = p_1, \\ \alpha_2 y(1) + \beta_2 y(0) + \gamma_2 \dfrac{dy(1)}{dx} + \delta_2 \dfrac{dy(0)}{dx} = p_2, \end{cases} \tag{5}$$

admits a unique solution in the form:

$$\begin{cases} y(x) = y_0(x) + p_1 \left(\dfrac{H_{01}}{\Delta} y_1(x) - \dfrac{H_{02}}{\Delta} y_2(x) + \displaystyle\sum_{i=1}^{k} \dfrac{H_{(s+i)2}}{\Delta} \widetilde{H_i}(x) + \sum_{j=1}^{s} \dfrac{H_{j2}}{\Delta} \widetilde{Q_j}(x) \right) + \\ p_2 \left(-\dfrac{H_{10}}{\Delta} y_1(x) - \dfrac{H_{20}}{\Delta} y_2(x) + \displaystyle\sum_{i=1}^{k} \dfrac{H_{(s+i)3}}{\Delta} \widetilde{H_i}(x) + \sum_{j=1}^{s} \dfrac{H_{j3}}{\Delta} \widetilde{Q_j}(x) \right) \\ \mu_j = \mu_j^0 + p_1 \dfrac{H_{j2}}{\Delta} + p_2 \dfrac{H_{j3}}{\Delta}, j = 1, \dots, s. \end{cases} \tag{6}$$

The entities denoted by H_{01}, H_{02}, H_{10}, $H_{20}, H_{(s+i)2}, H_{(s+i)3}, H_{j2}, H_{j3}$ constitute a collection of determinant expressions, the explicit constructions of which are expounded in Appendix 3. $y_0(x), \mu_j^0$ are identified as a solution to the problem encoded by the (1), (3).

Henceforth, we shall demonstrate how the solution to the dilemma hitherto propounded may be deduced from Theorem 2. To this end, it suffices to postulate that the $\vec{p}$ exhibits linearly and continuously dependence with respect to the prescribed collection $\{f(x)\in L_2(0,1),\omega_1\in\mathbb{C},\dots,\omega_s\in\mathbb{C}\}$. Stated otherwise, the components of the $\vec{p}$ manifest as linear continuous functionals on the space $L_2(0,1)\times\mathbb{C}^s$. By invoking the theorem characterizing the general structure of bounded linear functionals over the product space $L_2(0,1)\times\mathbb{C}^s$, we are thereby positioned to articulate the principal proposition of the present manuscript.

Theorem 3. Presume that $\sigma_1(x),\sigma_2(x)$ are arbitrary functions satisfying $\sigma_1(x)\in L_2(0,1)$, $\sigma_2(x)\in L_2(0,1)$, and let the $\vec{\xi_1}\in\mathbb{C}^s,\vec{\xi_2}\in\mathbb{C}^s$ be prescribed a priori. Then, under the non-vanishing condition $\Delta=\det\begin{pmatrix}A\\C_0\end{pmatrix}\neq 0$, the (1), subject to the supplementary conditions

$$\begin{cases}\alpha_1 y(1)+\beta_1 y(0)+\gamma_1\dfrac{dy(1)}{dx}+\delta_1\dfrac{dy(0)}{dx}-\displaystyle\int_0^1 l(y,\vec{\mu})\sigma_1(x)dx-\sum_{m=1}^{s}\theta_m(y,\vec{\mu})\xi_{1m}=0,\\ \alpha_2 y(1)+\beta_2 y(0)+\gamma_2\dfrac{dy(1)}{dx}+\delta_2\dfrac{dy(0)}{dx}-\displaystyle\int_0^1 l(y,\vec{\mu})\sigma_2(x)dx-\sum_{m=1}^{s}\theta_m(y,\vec{\mu})\xi_{2m}=0,\end{cases}\tag{7}$$

admits a unique solution for every datum $f(x)\in L_2(0,1),\omega_1\in\mathbb{C},\dots,\omega_s\in\mathbb{C}$, and this solution is furnished by the rendering given in

$$\begin{cases} y(x)=y_0(x)+\displaystyle\sum_{m=1}^{s}\omega_m\Bigg(\frac{H_{01}\xi_{1m}+H_{10}\xi_{2m}}{\Delta}y_1(x)+\frac{H_{02}\xi_{1m}+H_{20}\xi_{2m}}{\Delta}y_2(x)+\\ \displaystyle\sum_{i=1}^{k}\frac{H_{(s+i)2}\xi_{1m}+H_{(s+i)3}\xi_{2m}}{\Delta}\widetilde{H_i}(x)+\sum_{j=1}^{s}\frac{H_{j2}\xi_{1m}+H_{j3}\xi_{2m}}{\Delta}\widetilde{Q_j}(x)\Bigg)+\\ \displaystyle\frac{1}{\Delta}\int_0^1\frac{\begin{vmatrix}y_1(t)&y_2(t)\\y_1(1)&y_2(1)\end{vmatrix}+\begin{vmatrix}y_1(t)&y_2(t)\\y_1'(1)&y_2'(1)\end{vmatrix}}{\begin{vmatrix}y_1(t)&y_2(t)\\y_1'(t)&y_2'(t)\end{vmatrix}}f(t)dt\big(y_1(x)\big(H_{01}\sigma_1(t)+H_{10}\sigma_2(t)\big)+y_2(x)\big(H_{02}\sigma_1(t)+H_{20}\sigma_2(t)\big)+\\ \displaystyle\big(H_{(s+i)2}\sigma_1(t)+H_{(s+i)3}\sigma_2(t)\big)\sum_{i=1}^{k}\widetilde{H_i}(x)+\sum_{j=1}^{s}\big(H_{j2}\sigma_1(t)+H_{j3}\sigma_2(t)\big)\widetilde{Q_j}(x)\Bigg),\\ \mu_j=\mu_j^0+\displaystyle\frac{1}{\Delta}\left(\sum_{m=1}^{s}\omega_m\big(H_{j2}\xi_{1m}+H_{j3}\xi_{2m}\big)+\big(H_{j2}\sigma_1(t)+H_{j3}\sigma_2(t)\big)\int_0^1\frac{\begin{vmatrix}y_1(t)&y_2(t)\\y_1(1)&y_2(1)\end{vmatrix}+\begin{vmatrix}y_1(t)&y_2(t)\\y_1'(1)&y_2'(1)\end{vmatrix}}{\begin{vmatrix}y_1(t)&y_2(t)\\y_1'(t)&y_2'(t)\end{vmatrix}}f(t)dt\right).\end{cases}\tag{8}$$

Hence, Theorem 3 delineates the supplementary linear conditions (7) that guarantee the uniqueness of the solution. Furthermore, Theorem 3 is biconditional: the supplementary linear conditions therein are, in a certain regard, the most general admissible.

Theorem 4. Suppose that the BVP constituted by system (1), when supplemented by a certain assemblage of auxiliary linear conditions, admits a unique solution for every datum $f(x) \in L_2(0,1)$, $\omega_1 \in \mathbb{C}, \dots, \omega_s \in \mathbb{C}$. Then those auxiliary linear conditions are tantamount–up to linear equivalence–to conditions of the form (7), for suitably chosen functions $\sigma_1(x) \in L_2(0,1)$, $\sigma_2(x) \in L_2(0,1)$, $\overrightarrow{\xi_1} \in \mathbb{C}^s, \overrightarrow{\xi_2} \in \mathbb{C}^s$.

Proof of Theorem 1. To commence the demonstration of Theorem 1, we shall first introduce the requisite notational framework:

$$H_i(x) = \frac{dh_i}{dx}; \quad Q_j = \frac{dq_j(x)}{dx}.$$

Utilizing the aforementioned notational conventions, we recast the first equation of regimen (1) into the following form:

$$\frac{d^2 y}{dx^2} + r_1(x)\frac{dy}{dx} + r_0(x)y = F(x), \quad (9)$$

where, $F(x) = f(x) - \sum_{i=1}^{k} U_i(y)H_i(x) - \sum_{j=1}^{s} \mu_j Q_j(x)$. In order to establish Theorem 1, it suffices to substantiate the following assertion.

Lemma 1. Let $h_i(x) \in W_2^1[0,1]$ for $i = 1, \dots, k$ and $q_j(x) \in W_2^1[0,1]$ for $j = 1, \dots, s$, and let the $U_1(\cdot), \dots, U_k(\cdot)$ be chosen in the form of (2). Then, the general solution to equation (4) for any $f(x) \in L_2(0,1)$ is given by the following expression:

$$y(x) = \widetilde{C_1} y_1(x) + \widetilde{C_2} y_2(x) + \int_0^x \frac{\begin{vmatrix} y_1(t) & y_2(t) \\ y_1(x) & y_2(x) \end{vmatrix}}{\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}} F(t)dt \quad (10)$$

whereby $\widetilde{C_1}, \widetilde{C_2}$ are arbitrary constants.

In preparation for the verification of Lemma 1, we proceed by establishing a set of auxiliary notations. Specifically, we designate $y_1(x)$ and $y_2(x)$ as particular solutions to the corresponding homogeneous DE, each satisfying incipient data imposed at the $x = 0$.

$$\begin{aligned} & y''(x) + r_1(x)y'(x) + r_2(x)y(x) = 0, x \in (0,1), \\ & y_1(0) = 1, \qquad y_2(0) = 0, \\ & y_1'(0) = 0, \qquad y_2'(0) = 1. \end{aligned}$$

Proof of Lemma 1. Let us rigorously ascertain that the expression appearing on the right-hand side of identity (10) indeed satisfies the DE delineated in (9). To that end, observe the following:

$$y'(x) = \widetilde{C_1} y_1'(x) + \widetilde{C_2} y_2'(x) + \int_0^x \frac{\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(x) & y_2'(x) \end{vmatrix}}{\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}} F(t)dt,$$

$$y''(x) = \widetilde{C_1} y_1''(x) + \widetilde{C_2} y_2''(x) + F(x) + \int_0^x \frac{\begin{vmatrix} y_1(t) & y_2(t) \\ -r_1(x)y_1'(x) - r_0(x)y_1(x) & -r_1(x)y_2'(x) - r_0(x)y_2(x) \end{vmatrix}}{\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}} F(t)dt,$$

Consequently, one ascertains the validity of
$y''(x)+r_1(x)y'(x)+r_0(x)y(x)=F(x), \quad 0<x<1;$
within the established framework. Thus, the lemma stands established. It is worth observing that Theorem 1 ensues immediately as a corollary of Lemma 1. Proceeding further, by invoking the rendering (10), we determine the undetermined algebraic quantities $\mu_1,\ldots,\mu_s$ and $U_1(y),\ldots,U_k(y)$, and $\widetilde{C}_1,\widetilde{C}_2$. The extraction of these quantities is carried out via the employment of Cramer's method. By applying this technique, we first substitute the $\vec{\omega}$ into the j-th column of the $\begin{pmatrix} A \\ C_0 \end{pmatrix}$, followed by the substitution of the vector $\vec{b_1}$, provided that $\Delta=\det\begin{pmatrix} A \\ C_0 \end{pmatrix}\neq 0$. The $\vec{\omega}$ and $\vec{b_1}$ are expounded in detail in Appendix 1, for $j=1,\ldots,s$. To facilitate the extraction of the $U_1(y),\ldots,U_k(y)$ the subsequent procedure is undertaken, we first substitute the $\vec{\omega}$ into the $(s+i)-$th column of the $\begin{pmatrix} A \\ C_0 \end{pmatrix}$, followed by the substitution of the $\vec{b_1}$, provided that $\Delta=\det\begin{pmatrix} A \\ C_0 \end{pmatrix}\neq 0$, $i=1,\ldots,k$. Furthermore, in order to compute the $\widetilde{C}_1,\widetilde{C}_2$ one is required to successively substitute the $\vec{\omega}$ into the $(s+k+1)-$th and $(s+k+2)-$th columns of the $\begin{pmatrix} A \\ C_0 \end{pmatrix}$, followed by the insertion of the $\vec{b_1}$, contingent upon the non-vanishing of the $\Delta=\det\begin{pmatrix} A \\ C_0 \end{pmatrix}\neq 0$.

The verification of Theorem 2 fundamentally parallels the argumentative framework utilized in the establishment of Theorem 1.

Proof of Theorem 3. The proof of Theorem 3 follows from Theorem 2. To do this, it is sufficient to choose p_1 and p_2 as continuous linear functionals on the space $L_2(0,1)\times\mathbb{C}^s$. We shall now undertake the rigorous justification of Theorem 3. Then, according to the Riesz representation theorem on the general form of a bounded linear functional on $L_2(0,1)\times\mathbb{C}^s$, we obtain

$$\begin{cases} p_1(f,\vec{\omega})=\int_0^1 f(x)\overline{\sigma_1(x)}dx+\sum_{m=1}^{s}\omega_m\xi_{1m}, \\ p_2(f,\vec{\omega})=\int_0^1 f(x)\overline{\sigma_2(x)}dx+\sum_{m=1}^{s}\omega_m\xi_{2m}, \end{cases} \tag{11}$$

where $\sigma_1(x)\in L_2(0,1),\sigma_2(x)\in L_2(0,1)$,
$\vec{\xi_1}\in\mathbb{C}^s,\vec{\xi_2}\in\mathbb{C}^s$.

Now, let us replace $f(x)$ and $\omega_1,\ldots,\omega_s$, as defined by the system of equations (1), with the quantities $l(y,\vec{\mu})$ and $\theta_1(y,\vec{\mu}),\ldots,\theta_s(y,\vec{\mu})$ in relations (11). Then, relations (7) follow, which can be interpreted as new boundary conditions. Theorem 3 is thus completely proven.

Proof of Theorem 4. We now proceed to the demonstration of Theorem 4. Let us postulate that the original regimen of equations specified in (1) is augmented by a collection Σ of additional linear conditions, such that the extended system comprising the original relations and the supplementary conditions-admits a unique solution, expressed in the form

$$w=\left(y(x),x\in(0,1);\mu_1,\ldots,\mu_s\right)\in W_2^2(0,1)\times\mathbb{C}^s \tag{12}$$

for every admissible datum $f(x)\in L_2(0,1)$, $\omega_1\in\mathbb{C},\ldots,\omega_s\in\mathbb{C}$.

Let Λ_Σ denote the operator associated with the system of differential relations (1) augmented by the supplementary conditions Σ. By virtue of the embedding theorem, it follows that both $y(x)$ and its $y'(x)$ exhibit absolute continuity over the closed interval $[0,1]$. Consequently, for any admissible data set $f(x)\in L_2(0,1),\omega_1\in\mathbb{C},\ldots,\omega_s\in\mathbb{C}$ the $y(0)$ и $y'(0)$, $\mu_j, j=1,\ldots,s,$ are uniquely determined. Therefore, the quantity $y(0)$ may be regarded as a linear functional over the product space $L_2(0,1)\times\mathbb{C}^s$, insofar as its value varies linearly with respect to the choice of input data $f(x)\in L_2(0,1),\omega_1\in\mathbb{C},\ldots,\omega_s\in\mathbb{C}$. In light of the hypotheses stipulated in Theorem 1, it ensues that the inverse operator Λ_{Σ}^{-1} is bounded on the $L_2(0,1)\times\mathbb{C}^s$. As a consequence, the priori estimates encapsulated in relation

$$|y(0)|+|y'(0)|+\sum_{j=1}^{s}|\mu_j|\le C\|f\|_{L_2(0,1)}+\sqrt{\sum_{j=1}^{s}|\omega_j|^2} \tag{13}$$

are satisfied.

Consequently, from the a priori estimate (13), it follows that the functionals $V_1(y,\vec{\mu}),V_2(y,\vec{\mu})$ manifest as linear and bounded functionals defined on the $L_2(0,1)\times\mathbb{C}^s$. Invoking the classical Riesz rendering theorem concerning the canonical structure of bounded linear functionals in $L_2(0,1)\times\mathbb{C}^s$, one deduces the existence of a unique pair–specifically, a function $\sigma_1(x)\in L_2(0,1)$ and $\vec{\xi_1}\in\mathbb{C}^s$ such that the following rendering holds:

$$\begin{cases} V_1(y,\vec{\mu})=\int_0^1 f(x)\overline{\sigma_1(x)}dx+\sum_{m=1}^{s}\omega_m\xi_{1m}, \\ V_2(y,\vec{\mu})=\int_0^1 f(x)\overline{\sigma_2(x)}dx+\sum_{m=1}^{s}\omega_m\xi_{2m}, \end{cases} \tag{14}$$

where the construction of $V_2(y,\vec{\mu})$ proceeds in complete analogy with that of $V_1(y,\vec{\mu})$.

We now consider (1) augmented by the conditions (14), after substituting $f(x)$ with $l(y,\vec{\mu})$, and replacing $\omega_1,\dots,\omega_s$ with $\theta_1(y,\vec{\mu}),\dots,\theta_s(y,\vec{\mu})$. According to Theorem 4, this modified problem admits a unique solution, which coincides with the solution w as described in (12). Hence, it follows that the conditions specified in (14) are equivalent to the supplementary conditions Σ. The proof of Theorem 4 is thereby completed.

Appendix 1. In the context of Theorem 1, the following notations are introduced:

$$A_1=\left(-\sum_{m=1}^{s}(b_m+d_m)H_{4m}-(\alpha_1+\gamma_1)H_{04}+(\alpha_2+\gamma_2)H_{40}+\sum_{i=1}^{k}(\beta_{i1}+\beta_{i2})H_{i4}\right);$$

$$A_2=\left(-\sum_{m=1}^{s}(b_m+d_m)H_{5m}+(\alpha_1+\gamma_1)H_{05}-(\alpha_2+\gamma_2)H_{50}+\sum_{i=1}^{k}(\beta_{i1}+\beta_{i2})H_{i5}\right);$$

$$A_3=\left(-\sum_{m=1}^{s}(b_m+d_m)H_{(s+i)m}-(\alpha_1+\gamma_1)H_{0(s+i)}+(\alpha_2+\gamma_2)H_{(s+i)0}+\sum_{i_1=1}^{k}(\beta_{i_1 1}+\beta_{i_1 2})H_{i_1(s+i)}\right);$$

$$A_4=\left(-\sum_{m=1}^{s}(b_m+d_m)H_{jm}-(\alpha_1+\gamma_1)H_{0j}+(\alpha_2+\gamma_2)H_{j0}+\sum_{i=1}^{k}(\beta_{i1}+\beta_{i2})H_{ij}\right);$$

$$H_{s+i}(\vec{\omega})=\sum_{m=1}^{s}\omega_m H_{(s+i)m};i=1,\dots,k;$$

$$H_j(\vec{\omega})=\sum_{m=1}^{s}\omega_m H_{jm};j=1,\dots,s;$$

$$H_0(\vec{\omega})=\sum_{m=1}^{s}\omega_m H_{m1};$$

$$H_{11}(\vec{\omega})=\sum_{m=1}^{s}\omega_m H_{m2}.$$

whereby, $H_0(\vec{\omega})$ denotes the determinant of the matrix obtained by replacing the $(s+k+2)-$th column of the $\begin{pmatrix} A \\ C_0 \end{pmatrix}$ with the $\vec{\omega}$, and $H_{11}(\vec{\omega})$ represents the determinant of the matrix derived by substituting the $(s+k+2)-$th column of the $\begin{pmatrix} A \\ C_0 \end{pmatrix}$ with the $\vec{\omega}$, $H_{(s+i)}(\vec{\omega})$ denotes the determinant of the matrix obtained by replacing the $(s+i)-$th column of the $\begin{pmatrix} A \\ C_0 \end{pmatrix}$ with the $\vec{\omega}$, and $H_j(\vec{\omega})$ represents the determinant of the matrix derived by substituting the $j-$th column of the $\begin{pmatrix} A \\ C_0 \end{pmatrix}$ with the $\vec{\omega}$,, provided that, $\Delta = \det\begin{pmatrix} A \\ C_0 \end{pmatrix} \neq 0$.

$H_4(\tilde{f})$ denotes the determinant of the matrix obtained by replacing the $(s+k+1)-$ th column of the $\begin{pmatrix} A \\ C_0 \end{pmatrix}$ with the $\vec{b_1}$, and $H_5(\tilde{f})$ represents the determinant of the matrix derived by substituting the $(s+k+2)-$ th column of the $\begin{pmatrix} A \\ C_0 \end{pmatrix}$ with the $\vec{b_1}$, $H_{s+i}(\tilde{f})$ denotes the determinant of the matrix obtained by replacing the $(s+i)-$ th column of the $\begin{pmatrix} A \\ C_0 \end{pmatrix}$ with the $\vec{b_1}$, and $H_j(\tilde{f})$ represents the determinant of the matrix derived by substituting the $j-$ th column of the $\begin{pmatrix} A \\ C_0 \end{pmatrix}$ with the $\vec{b_1}$, provided that, $\Delta = \det\begin{pmatrix} A \\ C_0 \end{pmatrix} \neq 0$;

the H_{m1} emerges in the cofactor expansion of the $H_0(\vec{\omega})$ along its $(s+k+1)-$ th column, $m=1,\dots,s$;

the H_{m2} emerges in the cofactor expansion of the $H_{11}(\vec{\omega})$ along its $(s+k+2)-$ th column, $m=1,\dots,s$;

the $H_{(s+i)m}$ emerges in the cofactor expansion of the $H_{(s+i)}(\vec{\omega})$ along its $(s+i)-$ th column, $m=1,\dots,s;\ i=1,\dots,k$;

the H_{jm} emerges in the cofactor expansion of the $H_j(\vec{\omega})$ along its $j-$ th column, $m=1,\dots,s;\ j=1,\dots,s$;

the $H_{4m}, H_{i4}, H_{04}, H_{40}$ emerge in the cofactor expansion of the $H_4(\tilde{f})$ along its $(s+k+1)-$ th column $m=1,\dots,s;\ i=1,\dots,k$;

the $H_{5m}, H_{i5}, H_{05}, H_{50}$ emerge in the cofactor expansion of the $H_5(\tilde{f})$ along its $(s+k+2)-$ th column $m=1,\dots,s;\ i=1,\dots,k$;

the $H_{(s+i)m}, H_{0(s+i)}, H_{(s+i)0}, H_{i_1(s+i)}$ emerge in the cofactor expansion of the $H_{s+i}(\tilde{f})$ along its $(s+i)-$ th column $m=1,\dots,s;\ i=1,\dots,k$;

the $H_{jm}, H_{ij}, H_{0j}, H_{j0}$ emerge in the cofactor expansion of the $H_j(\tilde{f})$ along its $j-$ th column, $m=1,\dots,s;\ i=1,\dots,k$;

$$\vec{b_1} = \begin{bmatrix} -V_1(\tilde{f}) \\ \vdots \\ -V_1(\tilde{f}) \\ -U_1(\tilde{f}) \\ \vdots \\ -U_k(\tilde{f}) \\ -V_2(\tilde{f}) \\ -V_3(\tilde{f}) \end{bmatrix}; \quad \vec{\omega} = \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_s \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Appendix 2. Information about matrix $\begin{bmatrix} A \\ C_0 \end{bmatrix}$.

Let us recast (10) into the following equation:

$$y(x)=\widetilde{C_1}y_1(x)+\widetilde{C_2}y_2(x)+\widetilde{f}(x)-\sum_{i=1}^{k}U_i\widetilde{H_i}(x)-\sum_{j=1}^{s}\mu_j\widetilde{Q_j}(x), \tag{15}$$

hereby,

$$\widetilde{f}(x)=\int_0^x\frac{\begin{vmatrix}y_1(t) & y_2(t)\\ y_1(x) & y_2(x)\end{vmatrix}}{\begin{vmatrix}y_1(t) & y_2(t)\\ y_1'(t) & y_2'(t)\end{vmatrix}}f(t)dt;$$

$$\widetilde{H_i}(x)=\int_0^x\frac{\begin{vmatrix}y_1(t) & y_2(t)\\ y_1(x) & y_2(x)\end{vmatrix}}{\begin{vmatrix}y_1(t) & y_2(t)\\ y_1'(t) & y_2'(t)\end{vmatrix}}H_i(t)dt;$$

$$\widetilde{Q_j}(x)=\int_0^x\frac{\begin{vmatrix}y_1(t) & y_2(t)\\ y_1(x) & y_2(x)\end{vmatrix}}{\begin{vmatrix}y_1(t) & y_2(t)\\ y_1'(t) & y_2'(t)\end{vmatrix}}Q_j(t)dt.$$

By incorporating the second relation from the regimen (1), together with the subsidiary conditions (3), into equation (15), we obtain the following refined formulation.

$$\begin{cases}\sum_{j=1}^{s}a_{mj}\mu_j+\widetilde{C_1}V_1(y_1)+\widetilde{C_2}V_1(y_2)+V_1(\widetilde{f})-\sum_{i=1}^{k}U_i(y)V_1(\widetilde{H_i})-\sum_{j_1=1}^{s}\mu_{j_1}V_1(\widetilde{Q_{j_1}})=\omega_m,\ m=1,\ldots,s;\\ U_p(y)=\widetilde{C_1}U_p(y_1)+\widetilde{C_2}U_p(y_2)+U_p(\widetilde{f})-\sum_{i=1}^{k}U_i(y)U_p(\widetilde{H_i}(x))-\sum_{j_1=1}^{s}\mu_{j_1}U_p(\widetilde{Q_{j1}}(x)),\ p=1,\ldots k;\\ \widetilde{C_1}V_2(y_1)+\widetilde{C_2}V_2(y_2)+V_2(\widetilde{f})-\sum_{i=1}^{k}U_i(y)V_2(\widetilde{H_i})-\sum_{j_1=1}^{s}\mu_{j_1}V_2(\widetilde{Q_{j_1}})=0;\\ \widetilde{C_1}V_3(y_1)+\widetilde{C_2}V_3(y_2)+V_3(\widetilde{f})-\sum_{i=1}^{k}U_i(y)V_3(\widetilde{H_i})-\sum_{j_1=1}^{s}\mu_{j_1}V_3(\widetilde{Q_{j_1}})=0;\end{cases} \tag{16}$$

hereby,

$$V_1(y)=b_my(1)+c_my(0)+d_m\frac{dy(1)}{dx}+e_m\frac{dy(0)}{dx};$$

$$V_2(y)=\alpha_1y(1)+\beta_1y(0)+\gamma_1\frac{dy(1)}{dx}+\delta_1\frac{dy(0)}{dx};$$

$$V_3(y)=\alpha_2y(1)+\beta_2y(0)+\gamma_2\frac{dy(1)}{dx}+\delta_2\frac{dy(0)}{dx}.$$

(16) shall be reformulated in a compact matrix-vector framework.

Appendix 3. Information about theorem 2.

$H_1(\vec{p})$ denotes the determinant of the matrix obtained by replacing the $(s+k+1)-$th column of the $\begin{pmatrix} A \\ C_0 \end{pmatrix}$ with the $\vec{p} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ p_1 \\ p_2 \end{bmatrix}$, and $H_2(\vec{p})$ denotes the determinant of the matrix obtained by replacing the $(s+k+1)-$th column of the $\begin{pmatrix} A \\ C_0 \end{pmatrix}$ with the $\vec{p}$, $H_{(s+i)}(\vec{p})$ denotes the determinant of the matrix obtained by replacing the $(s+i)-$ th column of the $\begin{pmatrix} A \\ C_0 \end{pmatrix}$ with the $\vec{p}$, and $H_j(\vec{p})$ denotes the determinant of the matrix obtained by replacing the $j-$ th column of the $\begin{pmatrix} A \\ C_0 \end{pmatrix}$ with the $\vec{p}$, $\Delta = \det\begin{pmatrix} A \\ C_0 \end{pmatrix} \neq 0$. We shall henceforth refer to

$$H_1(\vec{p}) = p_1 \cdot H_{01} + p_2 \cdot H_{10};$$
$$H_2(\vec{p}) = p_1 \cdot H_{02} + p_2 \cdot H_{20};$$
$$H_{(s+i)}(\vec{p}) = p_1 \cdot H_{(s+i)2} + p_2 \cdot H_{(s+i)3}, i = 1,\ldots,k;$$
$$H_j(\vec{p}) = p_1 \cdot H_{j2} + p_2 \cdot H_{j3};\ j = 1,\ldots,s.$$

Conclusion

In the present manuscript, a hybrid-structured DE is examined. The principal aim is to explore the possibility of substituting the given supplementary boundary conditions with alternative, equivalent ones. This objective is achieved through the formulation and proof of four theorems, providing a rigorous foundation for the proposed substitutions. Incipiently, the existing boundary conditions are considered in a nonhomogeneous context. Subsequently, new boundary conditions, denoted as (7), are introduced. These newly formulated conditions are demonstrated to be equivalent to the original ones, ensuring the unique solvability of the hybrid-structured regimen labeled as (1). The regimen under consideration is characterized as hybrid due to the presence of both unknown functions $U_1(\cdot),\ldots,U_k(\cdot)$ and algebraic components. This dual nature necessitates a nuanced approach to boundary condition formulation and analysis. The methodology employed underscores the importance of flexibility in boundary condition specification, particularly in complex or hybrid regimens. By establishing the equivalence of different boundary conditions, the manuscript provides valuable insights into the solvability and analysis of such regimens. The results presented pertain to the smooth case, specifically when $h_i(x) \in W_2^1[0,1]$ for $i = 1,\ldots,k$ and $q_j(x) \in W_2^1[0,1]$ for $j = 1,\ldots,s$, with the $U_1(\cdot),\ldots,U_k(\cdot)$ selected as in equation (2). However, the scenario involving non - smooth sets of functions $\{h_1(\cdot),\ldots,h_k(\cdot)\}$ and $\{q_1(\cdot),\ldots,q_s(\cdot)\}$ presents significant interest. In such cases, the concept of a quasi - derivative becomes pertinent. This notion is extensively applied in monographs [27, 28]. The authors intend to dedicate a separate study to this particular case involving non - smooth functions $\{h_1(\cdot),\ldots,h_k(\cdot)\}$ and $\{q_1(\cdot),\ldots,q_s(\cdot)\}$.

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