IRSTI 27.03.66 https://doi.org/10.26577/ijmph.20251618

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(Received 17 March 2025; revised 10 April 2025; accepted 08 May 2025)

Pure Linear Orderings of Morley o-rank 1

Abstract. The study of pure linear orderings–sets equipped solely with a linear (total) order–has deep historical roots in mathematical logic and order theory. Initial investigations trace back to Cantor's work on ordinal numbers, which laid the foundation for understanding different sizes of ordered sets. Early 20th-century research by Hausdorff and others explored order types and their classification. Later, developments in model theory and set theory refined the structural properties of pure linear orderings, including their rigidity, embeddability, and definability in various logical frameworks. So, pure linear ordering and its classification are one of the classical mathematical questions. Descriptions of o-minimal and weakly o-minimal pure linear orderings are known, as well as we know that any pure linear ordering has an o-superstable elementary theory. The aim of this paper is to start investigation of pure linear orderings that have an o-ω-stable elementary theory. So, we give the complete description of pure linear ordering of Morley o-rank 1.
Key words: Pure linear ordering, o-minimal structure, o-stable theory, Dedekind's cut, ordered structure, Morley o-rank.

Introduction

A pure linear ordering is a set equipped solely with a total order, without additional algebraic or topological structure. The study of such orderings dates back to foundational work in set theory and logic, exploring their classification and embedding properties. A particularly important subclass is o-minimal orderings, where every definable subset consists of a finite union of points and intervals. These structures provide a well-behaved framework in model theory, ensuring tame topological and combinatorial properties, making them crucial in applications to real algebraic and analytic geometry. First, o-minimal pure linear orderings were classified, followed by weakly o-minimal ones. In this paper, we propose the description of pure linear orderings of Morley o-rank 1 and o-degree at most 2.

Literature review

Investigation of pure linear orderings is a classical mathematical question with a long bibliography that lasts till our days; for instance, one can see [1, 3–9, 17–18]. A. Pillay and C. Steinhorn described pure linear orderings that are o-minimal in [12], and B. Kulpeshov described such orderings that are weakly o-minimal in [10]. In [11], B. Kulpeshov and S. Sudoplatov described all possibilities of rigidity degrees for countable pure linear orderings, and in [12], they described all possibilities of rigidity degrees for weakly o-minimal pure linear orderings. Later, B. Baizhanov and V. Verbovskiy introduced the notion of an o-stable theory in [2] and proved that the elementary theory of any pure linear ordering is o-superstable. O-stable theories and their applications to the investigation of ordered groups and fields were investigated in [15–16]. V. Verbovskiy introduced the notion of the Morley o-rank and the Morley o-degree in 2008, one can find it in [15]. The concept of stability is too crude to describe pure linear orders because all of them are o-superstable. Still, the concept of the Morley o-rank and the Morley o-degree is a sharper tool for describing pure linear orders. In [14], A. Pillay and C. Steinhorn showed that an o-minimal structure has the Morley o-rank 1, and the Morley o-degree 1 (but in other terms), and B. Kulpeshov proved that weakly o-minimal theories have the Morley o-rank 1, and Morley o-degree at most 2 [10]. It is quite straightforward to show the existence of an ordered structure of the Morley o-rank 1 and the Morley o-degree 2 that is not weakly o-minimal. Thus, the question of describing all pure linear orderings of the Morley o-rank 1 and the Morley o-degree 2 is open. Our article is devoted to answering this question.

Methodology

This paper investigates the class of all pure linear orderings. We apply the methods of Model Theory, namely, of the relative stability, o-stability, and Morley o-rank, as well as the classical methods of investigation of pure linear orderings.

Results and discussion

Definition 1 (B. Baizhanov, V. Verbovskiy [2])

1. An ordered structure $\mathcal{M}$ is *o-stable in* λ if for any $A \subseteq M$ with $|A| \leq \lambda$ and for any cut $\langle C, D\rangle$ in $\mathcal{M}$ there are at most λ 1-types over A which are consistent with the cut $\langle C, D\rangle$, i.e. $|S^1_{\langle C,D\rangle}(A)| \leq \lambda$.

2. A theory T is *o-stable in* λ if every model of T is. Sometimes we write T is o-λ-stable.

3. A theory T is *o-stable* if there exists an infinite cardinal λ in which T is o-stable.

Definition 2 Let $s(x)$ be a partial 1-type, and let $\varphi(x)$ and $\psi(x)$ be formulae. We say that $\varphi(x)$ and $\psi(x)$ are s -*inconsistent* if $s(x) \cup \{\varphi(x), \psi(x))\}$ is inconsistent.

Definition 3 (V. Verbovskiy, [15])

1. We say that the Morley o-rank of a formula $\phi(x)$ inside a cut $\langle C, D\rangle$ in $\mathcal{M}$ is equal to or greater than 1 and write o-$RM_{\langle C,D\rangle,\mathcal{M}}(\phi) \geq 1$ for this, if $\{\phi(x)\} \cup \langle C, D\rangle$ is consistent.

2. o-$RM_{\langle C,D\rangle,\mathcal{M}}(\phi) \geq \alpha + 1$ if there are infinitely many pairwise $\langle C, D\rangle$ -inconsistent formulae $\psi_i(x)$ such that $RM_{\langle C,D\rangle,\mathcal{M}}(\phi(x) \wedge \psi_i(x)) \geq \alpha$.

3. If α is a limit ordinal, then o-$RM_{\langle C,D\rangle,\mathcal{M}}(\phi) \geq \alpha$ if o-$RM_{\langle C,D\rangle,\mathcal{M}}(\phi) \geq \beta$ for all $\beta < \alpha$.

4. o-$RM_{\langle C,D\rangle,\mathcal{M}}(\phi) = \alpha$ if o-$RM_{\langle C,D\rangle,\mathcal{M}}(\phi) \geq \alpha$ and $\neg(o\text{-}RM_{\langle C,D\rangle,\mathcal{M}}(\phi) \geq \alpha + 1)$.

5. We define the Morley o-rank of a formula $\phi(x)$ inside $\mathcal{M}$ as follows:

$$o\text{-}RM_{\mathcal{M}}(\phi) = \\ = \sup\{o\text{-}RM_{\langle C,D\rangle,\mathcal{M}}(\phi) : \langle C, D\rangle \text{ is a cut in } \mathcal{M}\}.$$

6. We define the Morley o-rank of a formula $\phi(x)$ as follows:

$$o\text{-}RM(\phi) = \sup\{o\text{-}RM_{\langle C,D\rangle,\mathcal{M}}(\phi) : \mathcal{M} \vDash \\ \vDash T \text{ and } \langle C, D\rangle \text{ is a cut in } \mathcal{M}\}.$$

We define the Morley o-degree of a formula inside a cut

1. Let $\langle C, D\rangle$ be a cut in an ordered structure $\mathcal{M}$ and let o-$RM_{\langle C,D\rangle,\mathcal{M}}(\phi) = \alpha$ for some formula ϕ. We say that the Morley o-degree of $\phi(x)$ inside the given cut $\langle C, D\rangle$ in $\mathcal{M}$ is equal to n and we write o-$MD_{\langle C,D\rangle,\mathcal{M}}(\phi) = n$ for this, if there exist exactly n pairwise $\langle C, D\rangle$ -inconsistent formulae $\psi_i(x)$ such that $RM_{\langle C,D\rangle,\mathcal{M}}(\phi(x) \wedge \psi_i(x)) = \alpha$.

2. We define the Morley o-degree of a formula $\phi(x)$ inside $\mathcal{M}$ as follows:

$$o\text{-}RM_{\mathcal{M}}(\phi) = \sup\{o\text{-}RM_{\langle C,D\rangle,\mathcal{M}}(\phi) : \langle C, D\rangle \\ \text{ is a cut in } \mathcal{M} \text{ and } o\text{-}RM_{\langle C,D\rangle,\mathcal{M}}(\phi) = \alpha\}.$$

3. We define the Morley o-rank of a formula $\phi(x)$ as follows:

$$o\text{-}RM(\phi) = \sup\{o\text{-}RM_{\langle C,D\rangle,\mathcal{M}}(\phi) : \mathcal{M} \vDash \\ \vDash T,\ \langle C, D\rangle \text{ is a cut in } \mathcal{M}, \text{ and} \\ o\text{-}RM_{\langle C,D\rangle,\mathcal{M}}(\phi) = o\text{-}RM(\phi)\}.$$

We define the Morley o-rank of a type as the minimal Morley o-rank of a formula from this type.

We aim to completely describe pure linear orderings whose elementary theory has the Morley o-rank 1 and the Morley o-degree 2. First, we recall the complete description of pure linear orderings that are weakly o-minimal.

We first recall some standard notations, as they have been given in [3]. If M and N are linear orderings, then $M+N$ denotes the ordered sum of M and N. As usual, ω represents the ordering of the natural numbers, ω^* the reverse ordering on the natural numbers, and $\mathbb{Q}$ the ordering of the rational numbers. Let F be the set of all finite linear orderings, and

$$G=F\cup\{\omega,\omega^*,\omega+\omega^*,\omega^*+\omega,\mathbb{Q}\}$$

Also, let WO be the collection of all ordered sums of the form $C_1+\ldots+C_m$, where C_i is elementarily equivalent to some member of G for each $i\le m$.

Theorem 1 (B. Kulpeshov [10]) *Any weakly o-minimal structure $\mathcal{M}$ restricted to the signature $\{=,<\}$ is a member of WO, and conversely, the first-order theory of any member of WO is a weakly o-minimal theory of linear order.*

Let $\mathcal{M}=(M,<)$ be a linearly ordered structure. We recall the following formulae. A formula $S(x)$ saying that an element x has a successor we define as $S(x)=\exists y(x<y\wedge\neg\exists z(x<z<y))$. In a similar way, we define a formula $P(x)$ that says that an element x has a predecessor. We define a formula $D(x,y)$ saying that each element in some open interval, containing both x and y, has neither a successor nor a predecessor:

$$D(x,y)=\exists u\exists v(u<x<v\wedge u<y<v\wedge$$
$$\wedge\forall z(u\le z\le v\to(\neg S(z)\wedge\neg P(z))).$$

Also, we define a formula $\Delta(x,y)$ which says that each element between x and y has both a successor and a predecessor, x has a successor, and y has a predecessor:

$$\Delta(x,y)=x<y\wedge S(x)\wedge P(y)\wedge$$
$$\wedge\forall z(x<z<y\to(S(z)\wedge P(z))).$$

Now we define an equivalence relation E, whose classes are maximal convex sets, ordered either densely or discretely:

$$E(x,y)=D(x,y)\vee\Delta(x,y)\vee\Delta(y,x)\vee x=y.$$

It is just a routine to check the reflexivity, symmetry, and transitivity of E, as well as that each E-class is convex.

Observe that if an element has a predecessor but no successor, then this element is a maximal element in its E-class. Similarly, if an element has a successor but no predecessor, then this element is a minimal element in its E-class.

We say that an E-class is *open* if it has neither a minimal element nor a maximal one.

Lemma 1 *Let T be a theory expanding the theory of linearly ordered sets of the Morley o-rank 1 and the Morley o-degree n. Let E be the equivalence relation defined above. Then, the number of E-classes that are not open and of cardinality bigger than n is finite.*

Proof. First, we claim that if an E-class is not open, then it is discretely ordered. Without loss of generality, we may assume that an E-class contains a maximal element a. If $[a]_E$ consists just of a, then the order is trivially discrete. Otherwise, $[a]_E$ contains some $b<a$. If $\Delta(b,a)$ holds, then obviously the order is discrete. Assume that $D(b,a)$ hold, then by definition, there exist c and d with $c<b<a<d$ such that each element in $[c,d]$ has neither successor nor predecessor, so do a, for a contradiction.

Assume the contrary, that there exist infinitely many E-classes that are not open and whose cardinality is bigger than n. By Dirichlet's principle, there exist infinitely many E-classes with a minimal element or infinitely many E-classes with a maximal element, say, the first case holds. Also, we can find either an infinite increasing chain of such E-classes or an infinite decreasing one. Without loss of generality, we can assume that there exists an infinite increasing sequence $\langle a_k:k<\omega\rangle$ such that $\neg E(a_i,a_j)$ for each $i<j$, and each E-class of a_k contains a minimal element b_k and has a cardinality bigger than n.

Let $C=\{c\in M:c<a_k$ for some $k<\omega\}$ and $D=M\setminus C$. Then the formula

$$\varphi(x)=\exists y(E(x,y)\wedge$$
$$\wedge\forall z(z<y\rightarrow\neg E(z,x)\wedge\exists^{>n}tE(x,t))$$

is consistent with the cut $(C,D)=\{c<x<d:c\in C,\,d\in D\}$. Indeed, let $c\in C$ and $d\in D$. Then there exists k such that $c<a_k$. Also, there exist $c'\in(a_{k+2},a_{k+3})$. Since $d>c'>a_{k+2}$, the E-class of a_{k+1} is a subset of (c,d). Then $\varphi(\mathcal{M})\cap(c,d)\supseteq[a_{k+1}]_E\neq\varnothing$. Since any non-empty finite intersection of intervals is an interval, we obtain that $\varphi(x)$ is consistent with the cut (C,D).

Let $\varphi_i(x)$ say, that x is the i-th successor of the minimal element from the E-class of x (here, the 0-th successor of an y is y). Obviously, each $\varphi_i(x)$ is consistent with (C,D) and $\varphi_i(x)\wedge\varphi_j(x)$ is inconsistent for each $i<j$. Hence, the Morley o-degree of the cut (C,D) is at least $n+1$ for a contradiction.

Lemma 2 *A structure of the form* $\mathbb{Q}+\mathbb{Q}\times F_n$ *has Morley o-degree* $n+1$, *as well as the structures* $\mathbb{Q}\times F_n+\mathbb{Q}$, $\omega+\mathbb{Q}\times F_n$, $\omega^*+\omega+\mathbb{Q}\times F_n$, $\mathbb{Q}\times F_n+\omega^*$, *and* $\mathbb{Q}\times F_n+\omega^*+\omega$.

Proof. Let $C=\mathbb{Q}$ and $D=\mathbb{Q}\times F_n$. Let $q\in C$. Then $[q]_E=\mathbb{Q}$. Let $a\in\mathbb{Q}\times F_n$. Then $[a]_E$ is a finite order consisting of n elements. Let $\varphi_i(x)$ be defined as in Lemma 1

We consider the cut (C,D). Then $\{E(x,q)\}\cup\{c<x<d:c\in C,d\in D\}$ is consistent as well as $\{\neg E(x,q)\wedge\varphi_i(x)\}\cup(C<x<D)$. That is, there are at least $n+1$ extensions.

The other cases are similar.

Theorem 2 *Let* $\mathcal{A}_1=(A_1,<,\Sigma_1)$ *and* $\mathcal{A}_2=(A_2,<,\Sigma_2)$ *be two linear orders whose elementary theories admit quantifier elimination; either* $\mathcal{A}_1$ *does not have a maximal element or* $\mathcal{A}_2$ *does not have a minimal element; and each symbol in* Σ_i *is definable in* $\mathcal{A}_i$; *moreover, it is either a unary predicate, unless it is the equivalence relation* E *defined above, or the successor function; also* $\Sigma_1\cap\Sigma_2$ *contains* $\{=,<,E\}$. *We consider their ordered sum* $\mathcal{B}=\mathcal{A}_1+\mathcal{A}_2$ *in the signature* $\Sigma=\{<,P_1,P_2\}\cup\Sigma_1\cup\Sigma_2$, *where* P_i *names* A_i. *Then, the elementary theory* $Th(\mathcal{A}_1+\mathcal{A}_2,\Sigma_1\cup\Sigma_2)$ *admits quantifier elimination.*

Proof. We show that the elementary theory of $\mathcal{B}=(B,\Sigma)$, where $B=A_1+A_2$ and $\Sigma=\Sigma_1\cup\Sigma_2$, admits quantifier elimination. By Tarski's test, it is sufficient to eliminate the existential quantifier in each formula of the type $\theta=\exists x\wedge_i\varphi_i(x,\overline{y}_i)$, where φ_i is either an atomic formula or the negation of an atomic formula.

We separate atomic formulae of Σ_1 and Σ_2 in the following way. Let, for $i\in\{1,2\}$,

$$\psi_i(x,\overline{z}_1)=\bigwedge_i\{\varphi_i(x,\overline{y}_i):\varphi_i\in\Sigma_i\}.$$

Then

$$\exists x\bigwedge_i\varphi_i(x,\overline{y}_i)\Leftrightarrow\exists x(\psi_1(x,\overline{y}_1)\wedge\psi_2(x,\overline{y}_2)).$$

If both ψ_1 and ψ_2 contain some unary predicates, then θ is inconsistent because these unary predicates define subsets of A_1 and of A_2, but $A_1\cap A_2=\varnothing$. If θ contains at least several occurrences of E, we replace them in the following way. Since $E(x,y)\wedge E(x,z)$ is equivalent to $E(x,y)\wedge E(y,z)$, we may assume that θ contains at most one occurrence of $E(x,y)$. Each atomic formula μ that does not contain x can be omitted in the following sense: $\exists x(F(x)\wedge\mu)\Leftrightarrow(\exists xF(x))\wedge\mu$. After all these manipulations, we obtain a formula of either the signature Σ_1 or Σ_2.

If it is a formula of Σ_1, but not of Σ_2, then we use quantifier elimination of $Th(A_1,\Sigma_1)$. If it is a formula of Σ_2, but not of Σ_1, then we use quantifier elimination of $Th(A_2,\Sigma_2)$. Assume that it is a formula of $\Sigma_1\cap\Sigma_2$. Let $\theta=\exists x\chi(x)$. Then

$$\exists x\chi(x) \Leftrightarrow \exists x((\chi(x) \wedge P_1(x)) \vee (\chi(x) \wedge P_2(x))) \Leftrightarrow$$
$$\Leftrightarrow \exists x(\chi(x) \wedge P_1(x)) \vee \exists x(\chi(x) \wedge P_2(x)).$$

We obtain two formulae of Σ_1 and Σ_2, correspondingly. Then we proceed as above.

Note that the successor function $s(x)$ cannot map an element from A_1 to an element from A_2 because by the hypothesis of this theorem, either A_1 does not have a maximal element or A_2 does not have a minimal element.

Let $H=\{\omega, \omega^*, \omega+\omega^*, \omega^*+\omega, \mathbb{Q}\}$. We define $\mathbb{Q}_d$ as follows. We take $\mathbb{Q} \cup \sqrt{2}\cdot\mathbb{Q}$ with the ordering coming from $\mathbb{R}$ and replace each element of $\mathbb{Q}$ by $\mathbb{Q}$ and each element of $\sqrt{2}\cdot\mathbb{Q}$ by $\omega^*+\omega$ to obtain $\mathbb{Q}_d$. Let

$$\tilde{H}=\{X\times(\omega^*+\omega+\mathbb{Q}) : X \in H\} \cup$$
$$\cup \{X \times (\mathbb{Q}+\omega^*+\omega) : X \in H\}.$$

Let

$$G_{1,2}=G \cup \tilde{H} \cup \{\mathbb{Q}\times F_2, \mathbb{Q}_d\}.$$

Note that if we consider the quotient of an element of $\tilde{H}$ by the equivalence relation E then we obtain a structure of the form $X \times F_2$.

Also, let $\mathcal{O}_{1,2}$ be the collection of all ordered sums of the form $C_1+\ldots+C_m$, where C_i is elementarily equivalent to some member of $G_{1,2}$ for each $i \le m$, where there are no two consecutive elements of the form $\mathbb{Q}$ and Z where $Z \in \{\mathbb{Q}\times F_2, \mathbb{Q}_d\}$ as well as the following sums: $\omega+Z$, $\omega^*+\omega+Z$, and $Z+\omega^*$, $Z+\omega^*+\omega$; also, we exclude the following cases: let $Y \in \tilde{H}$ be such that Y/E does not have a minimal element; then we exclude summands of the form $\mathbb{Q}+Y$ as well as $\omega+Y$, $\omega^*+\omega+Y$; if Y/E does not have a maximal element then we exclude summands $Y+\mathbb{Q}$, $Y+\omega^*$, $Y+\omega^*+\omega$.

Theorem 3 *Any totally ordered structure $\mathcal{M}$, whose elementary theory has the Morley o-rank 1 and the Morley o-degree 2, restricted to the signature $\{=,<\}$ is a member of $\mathcal{O}_{1,2}$, and conversely, the first-order theory of any infinite member of $\mathcal{O}_{1,2}$ has the Morley o-rank 1 and the Morley o-degree at most 2.*

Proof. Let E be the equivalence relation defined above. By Lemma 1, it holds that E-classes consisting of F_n for $n>2$ cannot form a dense subset of $\mathcal{M}$; otherwise, we obtain Morley o-degree at least $n>2$. Assume that E-classes consisting of F_2 form a dense subset of $(a,b)/E$ for some interval (a,b). We claim that then this interval consists of E-classes, that consist of F_2. Indeed, we can have in this interval E-classes that are singletons, that is, F_1, but then each cut in this interval has 3 completions: E-class of x is a singleton; E-class of x consists of 2 elements and x is the first one; E-class of x consists of 2 elements and x is the second one. So, (a,b) is elementarily equivalent to $\mathbb{Q}\times F_2$.

By Lemma 1 we can have infinitely many classes of the form either $\mathbb{Q}$ or $\omega^*+\omega$. We consider the definable subset Ω of M consisting either of dense E-classes (elementarily equivalent to $\mathbb{Q}$) or of discrete open E-classes (elementarily equivalent to $\omega^*+\omega$). Then Ω/E can have at most two consecutive elements: it is either F_2, or does not contain a distinguishable element; so, it is either $\mathbb{Q}+\omega^*+\omega$, or $\omega^*+\omega+\mathbb{Q}$.

Hence, we obtain that each summand is elementarily equivalent to some of $G_{1,2}$.

For each element from $G_{1,2}$ it is well-known that its elementary theory admits quantifier elimination or the quantifier elimination theorem is simple and straightforward. So, it is just a routine to check that each element of $G_{1,2}$ has Morley o-rank 1 and Morley o-degree at most 2. Theorem 2 provides quantifier elimination for any finite sum of linear ordering from $G_{1,2}$. Quantifier elimination shows that any infinite member of $\mathcal{O}_{1,2}$ has Morley o-rank 1 and Morley o-degree at most 2.

Let $\mathcal{M}$ be an ordered structure and $F(x)$ a formula. We define the *convex hull* F^c of F as:

$$F^c(x)=\exists y \exists z(F(y) \wedge F(z) \wedge y \leq x \leq z).$$

Let p be a 1-type. We define the *convex hull* p^c of p as the following partial type:

$$p^c(x)=\{F^c(x) : F(x) \in p(x)\}.$$

Sometimes we call a type, which is the convex hull of some type, a convex type.

Definition 4 Let $\mathcal{M}$ be an ordered structure and $p \in S_1(M)$.

1. We say that the Morley c-rank of a formula $\phi(x)$ inside p^c is equal to or greater than 1 and write $RM_{p^c,\mathcal{M}}(\phi(x)) \geq 1$ if $\{\phi(x)\} \cup p^c(x)$ is consistent.

2. $RM_{p^c,\mathcal{M}}(\phi)(x) \geq \alpha+1$ if there are infinitely many pairwise inconsistent formulas $\psi_i(x)$ such that $RM_{p^c,\mathcal{M}}(\phi(x) \wedge \psi_i(x)) \geq \alpha$.

3. If α is a limit ordinal, then $RM_{p^c,\mathcal{M}}(\phi(x)) \geq \alpha$ if $RM_{p^c,\mathcal{M}}(\phi(x)) \geq \beta$ for all $\beta<\alpha$.

4. $RM_{p^c,\mathcal{M}}(\phi(x))=\alpha$ if $RM_{p^c,\mathcal{M}}(\phi(x)) \geq \alpha$ and $RM_{p^c,\mathcal{M}}(\phi(x)) \ngeq \alpha+1$.

Similarly, one can determine the Morley c-degree of a formula inside a cut. One can also determine the Morley c-rank of a type.

We say that the Morley convex-rank of a formula $\phi(x)$ inside $\mathcal{M}$ equals

$$c\text{-}RM_{\mathcal{M}}(\phi(x))=\sup\{RM_{p^c,\mathcal{M}}(\phi(x)) : p \in S_1(M)\}.$$

Definition 5 Let T be a complete theory, expanding the theory of a linearly ordered set.

We say that the Morley convex-rank of a formula $\phi(x)$ in T equals

$$c\text{-}RM(\phi(x))=\sup\{c-RM_{\mathcal{M}}(\phi(x)) : \mathcal{M} \models T\}.$$

Let us define $\mathcal{C}_{1,2}$ as the set of all finite ordered sums of the form $C_1+\ldots+C_m$, where each C_i is elementarily equivalent to some element of $G_{1,2}$.

As it was proved in [1] each cut can have at most two extensions up to the type that are the convex hull of some type. In Theorem 3 we count the number of completions of a cut taking into account completions of each convex type. In the next theorem we consider these convex types separately, that is why the description is simpler. The proof is similar to the proof of Theorem 3, that is why we omit it.

Theorem 4 *c-* $RM(\mathcal{M})=1$ *and c-* $DM(\mathcal{M}) \leq 2$ *if and only if* $\mathcal{M} \equiv \mathcal{N}$ *for some* $\mathcal{N} \in \mathcal{C}_{1,2}$.

Conclusion

In this paper, we have given a description of pure linear orderings of the Morley o-rank 1 and the Morley o-degree at most 2. It will be interesting to continue such kind of description and give the characterization of all pure linear ordering of the Morley o-rank 1 for each Morley o-degree.

Acknowledgments

The authors were supported by the grant AP22685272 of the SC of the MSHE of the Republic of Kazakhstan.

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Author Biographies

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