IRSTI 27.37.15: 27.35.63 https://doi.org/10.26577/ijmph.20251619

M.N. Kulbay

Al-Farabi Kazakh National University, Almaty, Kazakhstan
e-mail: makulbay@gmail.com
(Received 27 February 2025;accepted 2 June 2025)

Inverse Source Problem for advection-diffusion equation from boundary measured data

Abstract. The inverse problem for the advection-diffusion equation is considered in this paper. The study focuses on reconstructing a space-dependent source of a variable-coefficient advection-diffusion equation with separable sources from time-dependent temperature measurements at the right boundary of the domain. The Tikhonov regularization method is used to determine the space-dependent source function. These problems arise in various fields of science and engineering. The source term takes the form of separated variables, where one function describes the time evolution and the other represents the spatial distribution of some contaminant source. Such source terms also arise as control terms in the context of heat equations. Numerical experiments were conducted to demonstrate the accuracy and robustness of the proposed method. A non-iterative inversion algorithm is developed and numerically implemented for identifying the unknown space-dependent source. Consequently, identifying space-dependent or time-dependent sources is crucial in addressing environmental issues, which served as the motivation for proposing this problem.
Key words: Inverse source problem, advection-diffusion equation, space-dependent heat source, Dirichlet data

Introduction

In this paper we consider the inverse source problem of identifying the unknown space-dependent source $F(x)$,

$$\begin{cases} u_t + au_x - (\alpha(x)u(x))_x - F(x)H(t) = 0, & (x,t) \in \Omega_T := (0,l) \times (0,T]; \\ u(x,0) = 0, & x \in (0,l); \\ u(0,t) = 0, u_x(l,t) = 0, & t \in (0,T], \end{cases} \tag{1}$$

from the Dirichlet boundary measured data

$$g(t) = u(l,t), \qquad t \in (0,T]. \tag{2}$$

Direct and inverse boundary value problems for advection-diffusion equations are extensively applied in the mathematical modelling of natural phenomena and technological processes [1-17].

Analytical approaches for solving direct diffusion-advection problems have been intensively investigated [3-5]. On the other hand, the investigation of corresponding inverse source problems has garnered significant attention in recent years mainly due to their numerous practical applications, such as identifying pollution sources in an environmental medium ([4-10]).

In [10] an analytical method based on the quazi-reversibility method and the Fourier transform tool was developed to solve advection dispersion equation in rivers inversely in time. A two-stage numerical approach to solve the sparse initial source identification of a diffusion-advection equation has been discussed in [11]. An inverse problem related to a fractional diffusion equation is considered in [12,14]. The ill-posedness, existence and uniqueness of the inverse source problem of time fractional diffusion wave equation in a cylinder are proved in [12]. In [13] the study focused on the reconstruction of an unknown source term from a partial internal measured data. The considered ill-posed inverse problem is formulated as a minimization one and numerical reconstruction of the unknown source terms is investigating using an iterative process. Two inverse source problems, such as the recovery of space dependent source term and determination of time dependent source term are considered and existence, uniqueness and stability results are proved in [14].

The continuation inverse problem for a 3D steady-state diffusion model inside a cylindrical layered medium is considered in [15]. This model is used to describe the process of heat conduction and diffusion in protective coatings of gas and oil pipelines, the shell of reactors in the chemical industry and other industrial processes. In such cases, only the outer side of coverings is available for measurements, when it is necessary to recover the conditions inside the coverings in order to control the equipment state. The diffusion coefficient is supposed to be a piecewise constant function, depending on radius, Cauchy data are given on the outer boundary of the cylinder and temperature is recovered at the inner boundary of the cylinder.

Inverse source problems focus on reconstructing unknown sources from measurable boundary and/or final output data. These problems arise in various field of science and engineering. Significantly, the existence, uniqueness and conditional stability for inverse source problems for partial differential equations with variable coefficients have been rigorously analyzed in [9,18-23].

In [9], the authors investigate the reconstruction of a time-dependent source of a one-dimensional evolution linear advection–dispersion–reaction equation with spatially varying coefficients. Well-posedness of the inverse problems for finding the space dependent vector function $p(x)$ in general parabolic systems with vector sources of the form $F(x,t)=H(x,t)\,p(x)+q(x,t)$ was investigated in [19]. In [20], the authors proved that in the parabolic heat equation $u_t+Lu=F(x)$ subject to zero Dirichlet condition on the boundary Γ of the domain $\Omega\in R^n$, $n\ge 1$, initial condition $u(x,0)=u_0$ and final data over-determination $u(x,T)=u_T$, a time-independent heat source $F\in L^2(\Omega)$ can be uniquely retrieved if $u_0,\ u_T\in H_0^1(\Omega)\cap H^2(\Omega)$. An iterative algorithm based on a sequence of well-posed direct problems which are solved at each iteration step using the boundary element method was also developed for finding the source $F(x)$, $x\in\Omega$. However, this algorithm was implemented numerically only for the one-dimensional heat equation with $L=-\partial_{xx}$ and $\Omega=(0,\ell)$. Simultaneous identification of the pair $\langle F,T_0\rangle$ of source terms in the heat conduction equation $u_t=\left(\alpha(x)u_x\right)_x+F(x,t)$ from final data overdetermination u_T and the Robin condition $-\alpha(l)u_x(l,t)=v[u(l,t)-T_0(t)]$, was proposed in [18].

The source term in [22,24-26] takes the form of separated variables, where $F(x)$ and $H(t)$ describe the time evolution and the spatial distribution of some contaminant source, respectively. In particular, the heat process of radioactive decay is modelled by the equation $u_t=u_{xx}+F(x)H(t)$, where $H(t)=e^{-kt}$ and $k>0$ is the decay rate [22]. Such source terms also arise as control terms in the context of the heat equations. Consequently, identifying space dependent or time dependent sources is crucial in addressing environmental issues, which served as the motivation for proposing the following problem. Both types of problems have been extensively studied over the past decades, with the uniqueness of solutions already established in the existing literature. Inverse source problems involving multiplicatively separable sources for parabolic and hyperbolic equations with constant coefficients were initially investigated in [24-26].

Note that although inverse source problems for parabolic equations are usually moderately ill-posed [27], their degree of ill-posedness could be quite different [23,28]. The degree of ill-posedness of inverse source problems for the advection-diffusion equation was investigated numerically using singular value decomposition of the input-output operators in [28]. This analysis showed that the degree of ill-posedness of recovering $F(x)$ for constant coefficients is essentially higher than that of the inverse problem of recovering $F(x)$ from final data overdetermination.

Mathematical modelling of the direct problem

Let $T>0$, $a=const$, α be a twice piecewise continuously differentiable function on $[0,l]$ such that $\alpha(x)>0$ for all $x\in[0,l]$, and $g(t)\in H^1(0,T)$. Unless stated otherwise, the space and the time-dependent sources are bounded functions such that $F(x)\in L^2(0,l)$ and $H(t)\in L^2(0,T)$ respectively. Moreover, their Lebesgue measures are strictly positive, i.e. $\mu(\text{supp}(F))>0$ and $\mu(\text{supp}(H))>0$. For the given known sources $F(x)$ and $H(t)$, the problem (1) is referred to as the direct problem. Using separation of variables in (1), we look for a solution to this direct problem of the form:

$$u(x,t)=\sum_{k=1}^{\infty}u_k(t)X_k(x),\ (x,t)\in\Omega_T, \quad (3)$$

where the functions $X_k(x)$ are solutions of the following boundary eigenvalue problem:

$$\begin{aligned}&-(\alpha(x)X'(x))'+aX'(x)=\lambda X(x),\ x\in(0,l),\\&X(0)=X'(l)=0,\end{aligned} \quad (4)$$

corresponding to the eigenvalues λ_k, $k\geq 1$.

The differential operator $\mathbf{L}=-\frac{d}{dx}\alpha(x)\frac{d}{dx}+a\frac{d}{dx}$ in (4) is not self-adjoint since, in general, $a\neq\alpha'(x)$ for $x\in(0,l)$. However, multiplying through by the weight function $\sigma(x)=\frac{p(x)}{\alpha(x)}$, where $p(x)=\exp\left[-\int_0^x\frac{a-\alpha'(s)}{\alpha(s)}ds\right]$ is the integrating factor, the system (4) takes the following self-adjoint form:

$$\begin{aligned}&-(p(x)X'(x))'=\lambda\sigma(x)X(x),\ x\in(0,l),\\&X(0)=X'(l)=0.\end{aligned} \quad (5)$$

Since p, p' and σ are continuous on $[0,l]$, and $p(x)>0$ and $\sigma(x)>0$ on $[0,l]$, the boundary eigenvalue problem (5) is a regular Sturm-Liouville problem [29]. Therefore, the eigenvalues are real, positive, simple and can be ordered such that $0<\lambda_1<\lambda_2<\ldots$, with $\lim_{n\to\infty}\lambda_n=\infty$. In addition, the corresponding eigenfunctions form a complete orthogonal family of $L^2_\sigma(0,l)=$ $=\left\{f:[0,l]\to R:\int_0^l f^2(x)\sigma(x)dx<\infty\right\}$, i.e.

$$\langle X_k,X_j\rangle_{L^2_\sigma(0,l)}=$$
$$=\int_0^l X_k(x)X_j(x)\sigma(x)dx=\langle X_k,X_j\rangle_{L^2_\sigma(0,l)}\delta_{kj},$$

and can be normalized to yield a family $\{\chi_k\}_{k\geq1}$ of orthonormal functions in $L^2_\sigma(0,l)$ such that $\langle X_k,X_j\rangle_{L^2_\sigma(0,l)}=\delta_{kj}$.

Moreover, for each function $f\in L^2_\sigma(0,l)$, the series $\sum_{k=1}^{\infty}\langle f,\chi_k\rangle_{L^2_\sigma(0,l)}\chi_k(x)$ converges to f in $L^2_\sigma(0,l)$.

Let us now represent the source term $F(x)$ in terms of the normalized eigenfunctions $\{\chi_k\}_{k\geq1}$,

$$F(x)=\sum_{k=1}^{\infty}\langle F,\chi_k\rangle_{L^2_\sigma(0,l)}\chi_k(x). \quad (6)$$

REMARK. *If* $F\in H^1(0,l)$ *and satisfies the same boundary conditions, i.e.* $F(0)=F'(l)=0$, *then the series expansion (6) converges absolutely and uniformly to* F *in* $(0,l]$.

By formally substituting the series (3) into equation (4) and using the series expansion (6), we find that the coefficients $u_k(t)$ must satisfy the following initial value problem

$$\begin{aligned}&u_k'(t)+\lambda_k u_k(t)=\langle F,\chi_k\rangle_{L^2_\sigma(0,l)}H(t),\\&u_k(0)=0.\end{aligned} \quad (7)$$

Hence, the coefficients $u_k(t)$ of (3) can be uniquely determined

$$u_k(t)=\sum_{k=1}^{\infty}\langle F,\chi_k\rangle_{L^2_\sigma(0,l)}\int_0^t H(s)e^{-\lambda_k(t-s)}ds, \quad (8)$$

and the generalized solution of the direct problem (1) becomes

$$u(x,t)= \\ =\sum_{k=1}^{\infty}\langle F,\chi_k\rangle_{L^2_\sigma(0,l)}\chi_k(x)\int_0^t H(s)e^{-\lambda_k(t-s)}ds. \quad (9)$$

Reconstruction method for Inverse Source Problem

In the inverse source problem considered here, the space-dependent source $F(x)$ in the advection-diffusion equation (1) needs to be recovered from the time-dependent Dirichlet boundary data (2) supposing that the function $H(t)$ is known. As explained in [30], the solution to this inverse problem is unique only if additional constraints are imposed on the source function $F(x)$.

In order to reconstruct the unknown space-dependent source $F(x)$, we have to find the coefficients $F_k=\langle F,\chi_k\rangle_{L^2_\sigma(0,l)}$, $k\ge 1$, of the following series expansion

$$u(l,t)=\sum_{k=1}^{\infty}F_k\psi_k(t), \quad (10)$$

where

$$\psi_k(t)=\chi_k(l)\int_0^t H(s)e^{-\lambda_k(t-s)}ds,\ k\ge 1, \quad (11)$$

from boundary measured data (2). Since H is considered to be bounded on $[0,T]$, and the eigenfunctions χ_k and the coefficients F_k are also bounded, the terms of the series (10) in absolute value tend to zero faster than λ_k^{-1}, i.e.

$$|F_k\psi_k(t)|= \\ =|F_k\chi_k(l)|\left|\int_0^t H(t-s)e^{-\lambda_k s}ds\right|\le \\ \le C_5\int_0^t e^{-\lambda_k s}ds\le\frac{C_5}{\lambda_k}, k\ge 1.$$

Moreover, the eigenvalues λ_k for $k\ge 1$ are asymptotically quadratic with respect to k. Thence, we can truncate the series (10) and approximate $u(l,t)$ by the sum of a finite and sufficiently large number K of initial terms

$$u(l,t)\approx\sum_{k=1}^{K}F_k\psi_k(t). \quad (12)$$

Through this approach, we reduce the inverse problem to a finite dimensional one, specifically the identification of the K coefficients F_k in (12) based on the given boundary data (2). Consequently, Tikhonov regularization is employed, and we aim to determine the unrestricted minimum of the functional

$$J_\beta^{(1)}(F)=\frac{1}{2}\|u(l,t)-g^\varepsilon(t)\|^2_{L^2(0,T)}+\frac{\beta}{2}\|F\|_2^2= \\ =\frac{1}{2}\int_0^T\left(\sum_{k=1}^{K}F_k\Psi_k(t)-g(t)\right)^2dt+ \\ +\frac{\beta}{2}\sum_{k=1}^{K}F_k^2, \quad (13)$$

where β is the regularization parameter and it should be chosen as a good compromise between fitting the measured boundary data and guaranteeing the stability of the solution. To find the minimizer $F\in R^K$, we set all first partial derivatives of the functional (13) to zero, i.e.

$$\frac{\partial J_\beta^{(1)}}{\partial F_j}=\int_0^T\left(\sum_{k=1}^{K}F_k\Psi_k(t)-g(t)\right)\frac{\partial}{\partial F_j}\left(\sum_{k'=1}^{K}F_{k'}\Psi_{k'}(t)\right)dt+ \\ +\frac{\beta}{2}\frac{\partial}{\partial F_j}\left(\sum_{k=1}^{K}F_k^{\ 2}\right)=\int_0^T\left(\sum_{k=1}^{K}F_k\Psi_k(t)-g(t)\right)\Psi_j(t)dt+ \\ +\beta F_j=0, \quad j=1,\ldots,K.$$

Hence, the following necessary minimality conditions are obtained

$$\frac{\partial J_{\beta}^{(1)}}{\partial F_j} = \sum_{k=1}^{K} F_k \int_0^T \Psi_k(t)\Psi_j(t)\,dt - \\ -\int_0^T g(t)\Psi_j(t)\,dt + \beta F_j = 0,$$

for all $j = 1,\ldots,K$. (14)

The system of linear equations (14) can be expressed under the matrix form as

$$\left(\mathbf{A} + \beta \mathbf{I}_K\right)\mathbf{F}^{(\beta)} = \mathbf{b}^{\varepsilon}, \tag{15}$$

where the entries of the $K \times K$ matrix $\mathbf{A}$ and of the right-hand side vector $\mathrm{b} \in \mathrm{R}^K$ are given by

$$A_{jk} = \int_0^T \Psi_j(t)\Psi_k(t)\,dt,\ j = 1,\ldots,K,\ k = 1,\ldots,K, \tag{16}$$

$$b_j^{\varepsilon} = \int_0^T g^{\varepsilon}(t)\Psi_j(t)\,dt,\ j = 1,\ldots,K. \tag{17}$$

If $F^{(\beta)} \in \mathrm{R}^{\mathrm{K}}$ is the solution of equation (15) for an optimal value of the regularization parameter β, which can be determined by the *L*-curve criterion, then the approximate solution of recovered problem is

$$F(x) \approx \sum_{k=1}^{K} F_k^{(\beta)} \chi_k(x),\ x \in [0, L]. \tag{18}$$

Results and Discussion

The first numerical examples consider reconstructions of several smooth space-dependent sources for different given time-dependent sources, and linear and non-linear diffusion coefficients. In order to quantify the quality of reconstructed sources, F_{rec}, the following L^2-relative errors were computed in each case

$$\varepsilon_F = \frac{\left\|F(x) - F_{\mathrm{rec}}(x)\right\|_{L^2(0,1)}}{\left\|F(x)\right\|_{L^2(0,1)}}. \tag{19}$$

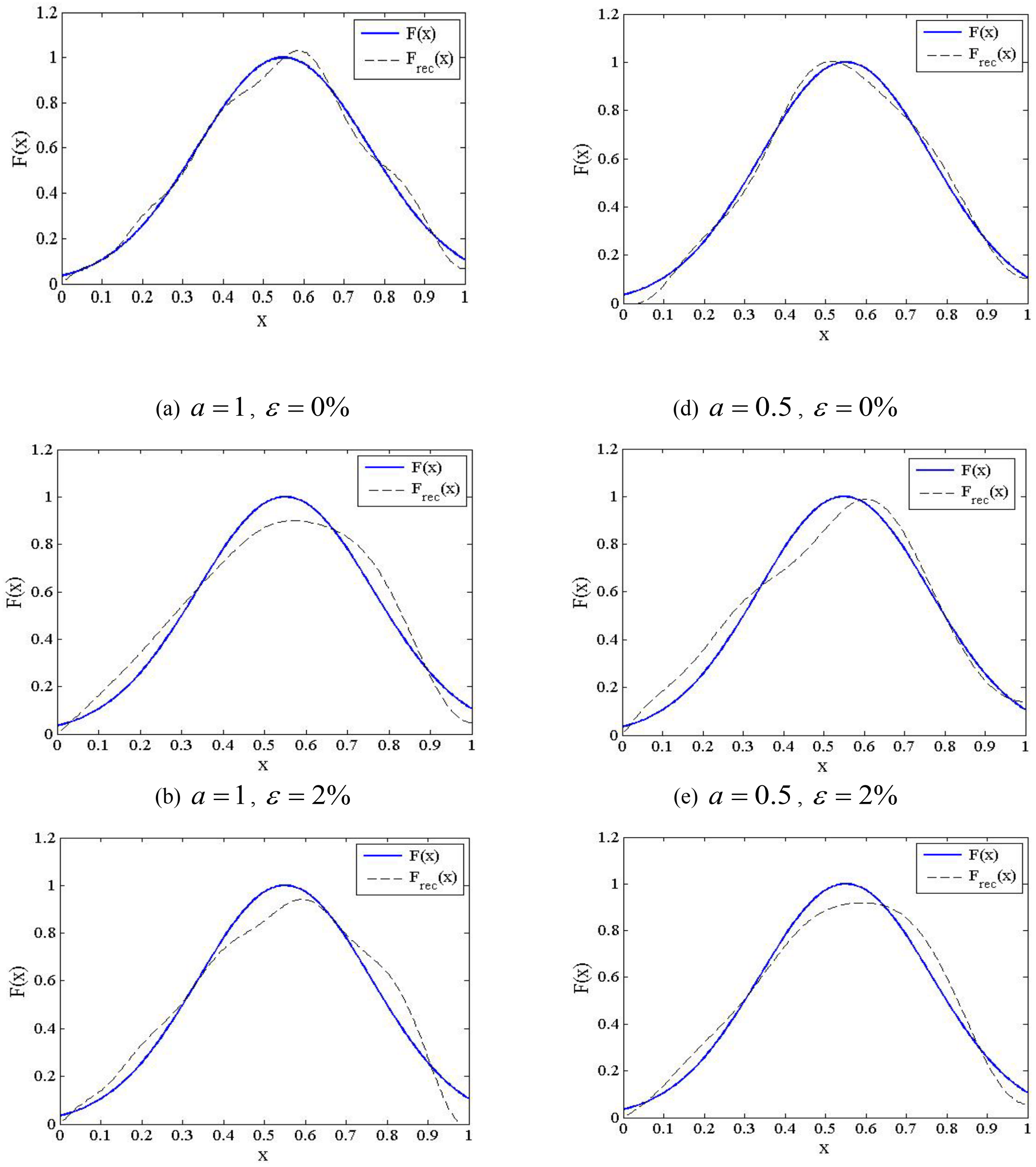
EXAMPLE 1. We attempt to reconstruct first smooth space-dependent sources of the form

$$F(x) = A_1 \exp\left(-\left(\frac{x - x_1}{\delta_1}\right)^2\right) + \\ + A_2 \exp\left(-\left(\frac{x - x_2}{\delta_2}\right)^2\right) \tag{20}$$

for a known time-dependent source $H(t) = \exp(-t)$. We consider both linear and non-linear diffusion coefficients with various parameter choices, as follows:

i) $A_1 = 1$, $x_1 = 0.55$, $\delta_1 = 0.3$, and $A_2 = 0$. The numerical reconstructions of the space-dependent source obtained in this case for $\alpha(x) = 0.5 + x$ and different choices of the coefficient a, $a = 1$ and $a = 0.5$ are given in Figure 1 and Table 1.

ii) $A_1 = 0.5$, $A_2 = 1$, $x_1 = 0.25$, $x_2 = 0.75$, $\delta_1 = \delta_2 = 0.11$ and $a = 1$. Reconstruction results for different expressions of the spatially varying diffusion coefficients, $\alpha(x) = 0.5 + x$ (linear) and are $\alpha(x) = 1 + x^2$ (non-linear), are presented in Figure 2 and Table 2.



(a) $a=1$, $\varepsilon=0\%$ (d) $a=0.5$, $\varepsilon=0\%$

(b) $a=1$, $\varepsilon=2\%$ (e) $a=0.5$, $\varepsilon=2\%$

(c) $a=1$, $\varepsilon=5\%$ (f) $a=0.5$, $\varepsilon=5\%$

Figure 1 – Reconstruction of the space-dependent source $F(x)$ for Example 1 i) and various noise levels in the data, $\varepsilon \in \{0,2,5\}\%$: $a=1$ in (a), (b) and (c); $a=0.5$ in (d), (e) and (f).

Table 1 – Regularization parameter values β and relative errors ε_F in the reconstruction of the space-dependent source $F(x)$ for Example 1 i) and various noise levels in the data, $\varepsilon \in \{0,2,5\}\%$

	$a=1$			$a=0.5$		
ε	0%	2%	5%	0%	2%	5%
β	$5\cdot 10^{-8}$	$3\cdot 10^{-7}$	$8\cdot 10^{-7}$	$3\cdot 10^{-8}$	$6\cdot 10^{-8}$	$5\cdot 10^{-7}$
ε_F	0.057	0.1161	0.12	0.0503	0.1101	0.119

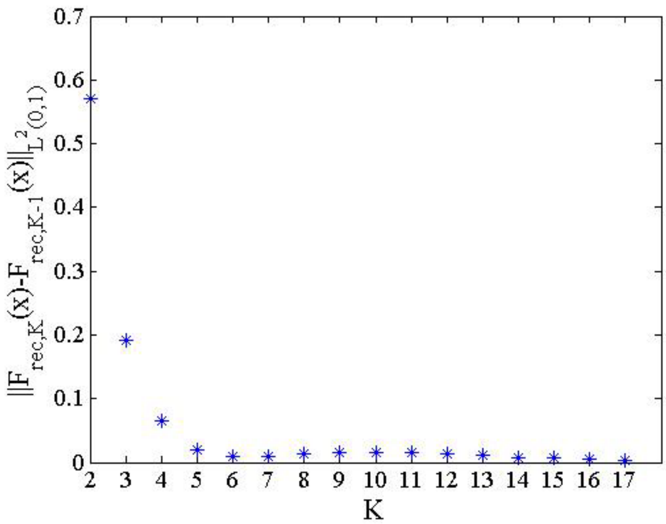
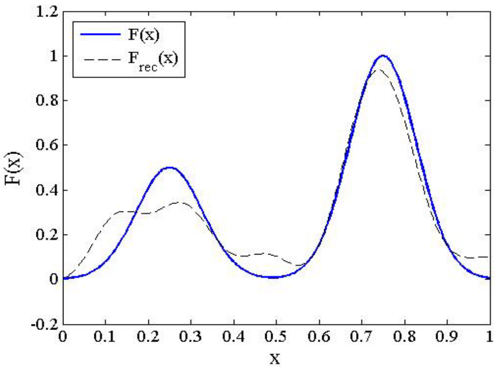
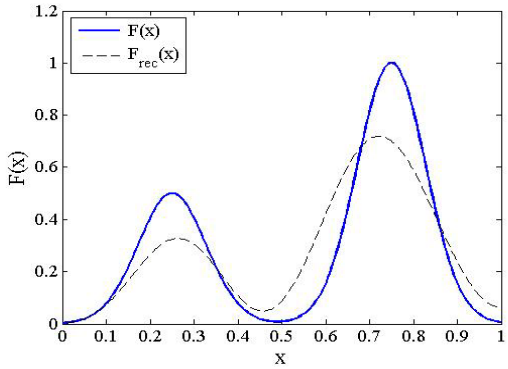


Figure 2 – The L^2-norm of the difference between the reconstructed space-dependent sources for K and $K-1$ eigenpairs used in numerical implementation, $F_{\text{rec},K}(x)$ and $F_{\text{rec},K-1}(x)$, respectively, for Example 1 i), 0% errors in the data and $a=1$.



(a) $\alpha(x)=0.5+x$



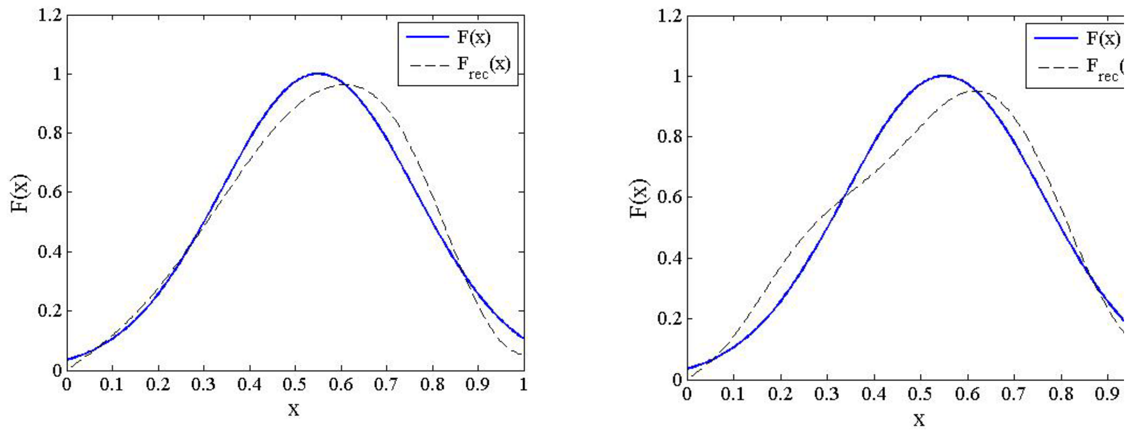
(b) $\alpha(x)=1+x^2$

Figure 3 – Reconstruction of the space-dependent source $F(x)$ for Example 1 ii) and data with 2% errors.

Table 2 – Regularization parameter values β and relative errors ε_F in the reconstruction of the space-dependent source $F(x)$ for Example 1 ii) and various noise levels in the data, $\varepsilon \in \{0,2,5\}\%$.

	$\alpha(x)=0.5+x$			$\alpha(x)=1+x^2$		
ε	0%	2%	5%	0%	2%	5%
β	$2\cdot10^{-9}$	$5\cdot10^{-8}$	$9\cdot10^{-8}$	$3\cdot10^{-8}$	$3\cdot10^{-8}$	$9\cdot10^{-8}$
ε_F	0.1830	0.2011	0.2694	0.3418	0.3681	0.3902

EXAMPLE 2. We reconstruct a smooth space-dependent source of the form (20) with the same parameter choices as in Example 1 i), i.e. $A_1=1$, $x_1=0.55$, $\delta_1=0.3$ and $A_2=0$, but in this case $H(t)=\sin(\pi t)$. Figure 4 and Table 3 contain the results of numerical reconstructions of the space-dependent source obtained for two different choices of the coefficient a, $a=1$ and $a=0.5$ and a linear diffusion coefficient, $\alpha(x)=0.5+x$. Results for a non-linear spatially varying diffusion coefficient, $\alpha(x)=1+x^2$, are found in Figure 5 and Table 4. A space-dependent source of the form (20) with the same parameter choices as in Example 1 ii) could not be recovered for the time-dependent source considered in this example.

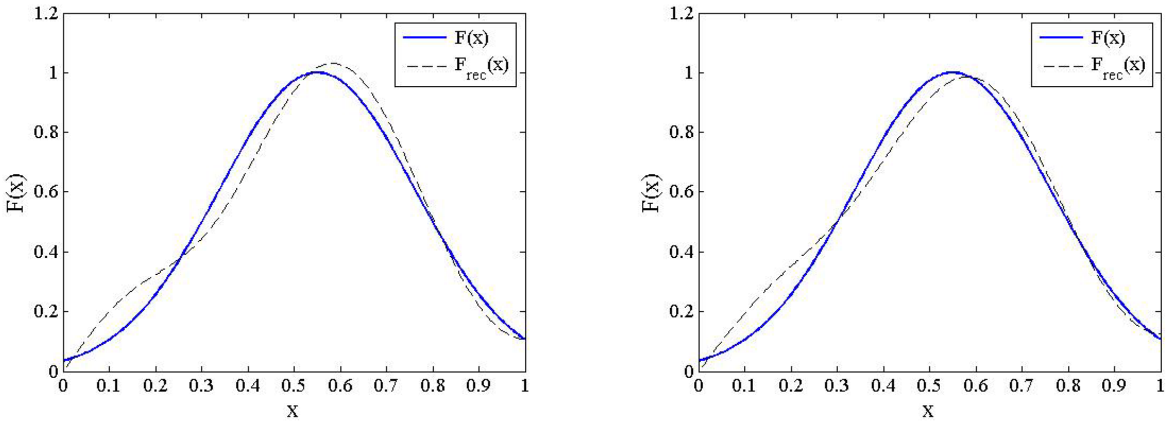


(a) $a=1$ (b) $a=0.5$

Figure 4 – Reconstruction of the space-dependent source $F(x)$ for Example 2, $\alpha(x)=0.5+x$, and data with 2% errors.

Table 3 – Regularization parameter values β and relative errors ε_F in the reconstruction of the space-dependent source $F(x)$ for Example 2, $\alpha(x)=0.5+x$ and various noise levels in the data, $\varepsilon \in \{0,2,5\}\%$.

	$a=1$			$a=0.5$		
ε	0%	2%	5%	0%	2%	5%
β	$9\cdot10^{-8}$	$4\cdot10^{-7}$	$5\cdot10^{-7}$	$9\cdot10^{-8}$	$3\cdot10^{-7}$	$5\cdot10^{-7}$
ε_F	0.063	0.092	0.0987	0.0667	0.0752	0.1378



(a) $a=1$ (b) $a=0.5$

Figure 5 – Reconstruction of the space-dependent source $F(x)$ for Example 2, $\alpha(x)=1+x^2$, and data with 2% errors.

Table 4 – Regularization parameter values β and relative errors ε_F in the reconstruction of the space-dependent source $F(x)$ for Example 2, $\alpha(x)=1+x^2$ and various noise levels in the data, $\varepsilon \in \{0,2,5\}\%$.

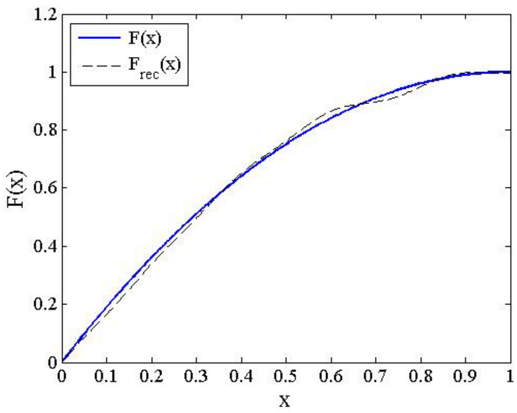
	$a=1$			$a=0.5$		
ε	0%	2%	5%	0%	2%	5%
β	$1\cdot 10^{-8}$	$1\cdot 10^{-8}$	$9\cdot 10^{-8}$	$1\cdot 10^{-8}$	$1\cdot 10^{-8}$	$5\cdot 10^{-8}$
ε_F	0.0804	0.0996	0.2142	0.1	0.0872	0.1639

EXAMPLE 3. We consider the following space-dependent sources which both satisfy the boundary conditions $F(0)=F'(1)=0$:

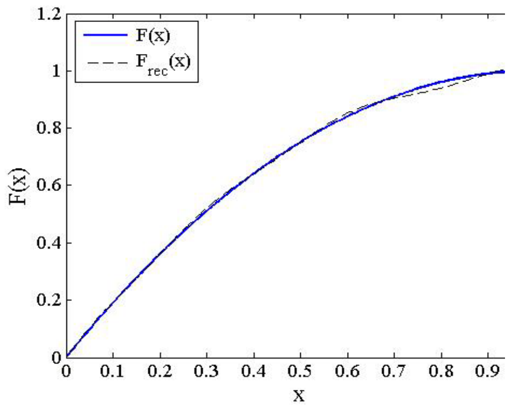
i) $F(x)=x(2-x)$, (21)

ii) $F(x)=\sin^2(\pi x)$. (22)

Reconstructions of the source in i) and two different known time-dependent sources, $H(t)=\exp(-t)$ and $H(t)=\sin(\pi t)$, are shown in Figure 6 and Table 5, and in Figure 7 and Table 6, respectively. The numerical recoveries of the source in ii) for $H(t)=\exp(-t)$ are presented in Figure 8 and Table 7. In all these numerical experiments, the diffusion coefficient was considered to be linear and of the form $\alpha(x)=0.5+x$.



(a) $a = 1$

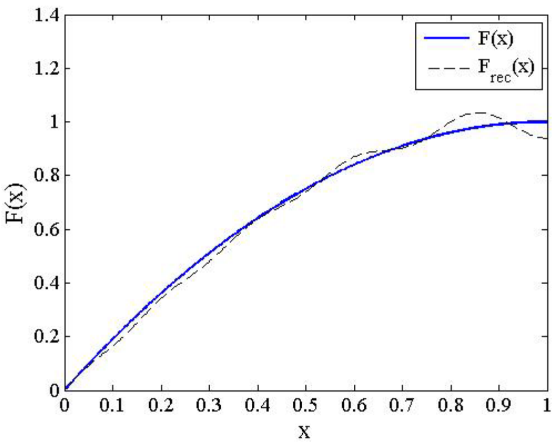


(b) $a = 0.5$

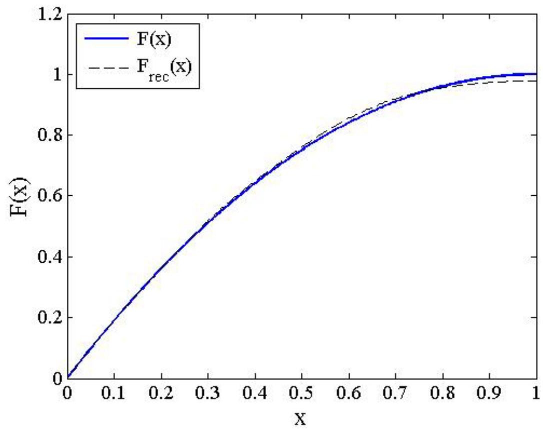
Figure 6 – Reconstruction of the space-dependent source $F(x)$ for Example 6 i), $\alpha(x) = 0.5 + x$, $H(t) = \exp(-t)$, and data with 2% errors.

Table 5 – Regularization parameter values β and relative errors ε_F in the reconstruction of the space-dependent source $F(x)$ for Example 3 i), $\alpha(x) = 0.5 + x$, $H(t) = \exp(-t)$, and various noise levels in the data, $\varepsilon \in \{0, 2, 5\}\%$.

	$a = 1$			$a = 0.5$		
ε	0%	2%	5%	0%	2%	5%
β	$9 \cdot 10^{-8}$	$1 \cdot 10^{-7}$	$3 \cdot 10^{-6}$	$5 \cdot 10^{-6}$	$5 \cdot 10^{-6}$	$9 \cdot 10^{-6}$
ε_F	0.02	0.0212	0.053	0.0068	0.0127	0.0173



(a) $a = 1$

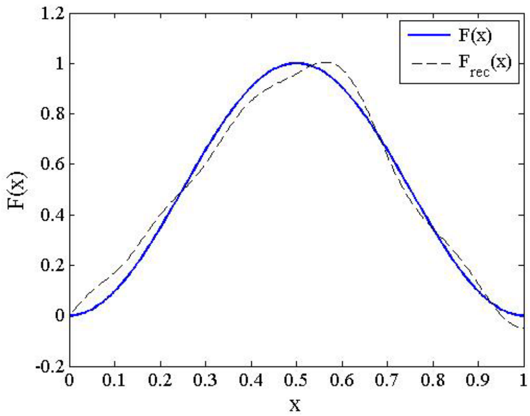


(b) $a = 0.5$

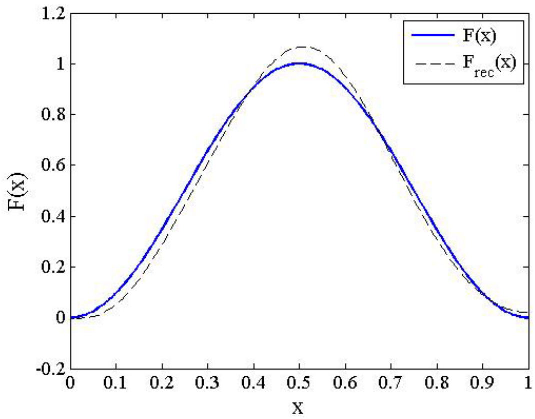
Figure 7 – Reconstruction of the space-dependent source $F(x)$ for Example 3 i), $\alpha(x) = 0.5 + x$, $H(t) = \sin(\pi t)$, and data with 2% errors.

Table 6 – Regularization parameter values β and relative errors ε_F in the reconstruction of the space-dependent source $F(x)$ for Example 3 i), $\alpha(x)=0.5+x$, $H(t)=\sin(\pi t)$, and various noise levels in the data, $\varepsilon\in\{0,2,5\}\%$.

	$a=1$			$a=0.5$		
ε	0%	2%	5%	0%	2%	5%
β	$5\cdot10^{-6}$	$5\cdot10^{-6}$	$5\cdot10^{-6}$	$5\cdot10^{-6}$	$5\cdot10^{-6}$	$5\cdot10^{-6}$
ε_F	0.0299	0.0352	0.0368	$\varepsilon_F=0.0115$	$\varepsilon_F=0.0292$	$\varepsilon_F=0.0208$



(a) $a=1$



(b) $a=0.5$

Figure 8 – Reconstruction of the space-dependent source $F(x)$

for Example 6 ii), $\alpha(x)=0.5+x$ $H(t)=\exp(-t)$, and data with 2% errors.

Table 7 – Regularization parameter values β and relative errors ε_F in the reconstruction of the space-dependent source $F(x)$ for Example 3 ii), $\alpha(x)=0.5+x$, $H(t)=\exp(-t)$, and various noise levels in the data, $\varepsilon\in\{0,2,5\}\%$.

	$a=1$			$a=0.5$		
ε	0%	2%	5%	0%	2%	5%
β	$5\cdot10^{-8}$	$5\cdot10^{-8}$	$5\cdot10^{-7}$	$2\cdot10^{-8}$	$2\cdot10^{-8}$	$1\cdot10^{-8}$
ε_F	0.0727	0.0793	0.1305	$\varepsilon_F=0.0463$	$\varepsilon_F=0.0652$	$\varepsilon_F=0.0910$

Conclusion

Overall, the reconstructions of space-dependent sources $F(x)$ obtained for a smaller value of the coefficient a (i.e. $a=0.5$) are comparable to, but slightly better than, the results obtained for the larger value $a=1$. Better and more stable source identifications were also achieved in the case of the linear diffusion coefficient $\alpha(x)=0.5+x$ and more importantly, in the case when the source term $F(x)$ satisfies the same boundary conditions as the eigenfunctions of the Sturm-Liouville problem (5) (i.e. $F(0)=F'(1)=0$). To conclude, our numerical experiments seem to suggest that the proposed reconstruction method works rather well for identifying smooth space-dependent sources albeit satisfying the natural boundary conditions $F(0)=F'(1)=0$ and $H(0)\neq 0$.

References

1 Alifanov O. *Inverse Heat Transfer Problems*. Berlin Heidelberg: Springer-Verlag, 1994.

2 Beck J., Blackwell B., and Clair C. *Inverse Heat Conduction: Ill-Posed Problems*. New York: Wiley, 1985.

3 Chen J., Liu C., and Liao C. "Two-Dimensional Laplace-Transformed Power Series Solution for Solute Transport in a Radially Convergent Flow Field." *Advances in Water Resources* 26 (2013): 1113–1124.

4 Fry V., Istok J., and Guenther R. "Analytical Solutions to the Solute Transport Equation with Rate-Limited Desorption and Decay." *Water Resources Research* 29 (1993): 3201–3208.

5 Neelz S. "Limitations of an Analytical Solution for Advection-Diffusion with Variable Coefficients." *Communications in Numerical Methods in Engineering* 22 (2006): 387–396.

6 Okubo A. *Diffusion and Ecological Problems: Mathematical Models*. New York: Springer, 1980.

7 Badia A.E., Duong T.H., and Hamdi A. "Identification of a Point Source in a Linear Advection-Dispersion-Reaction Equation: Application to a Pollution Source Problem." *Inverse Problems* 21 (2005): 1121–1136.

8 Hamdi A. "Inverse Source Problem in a 2D Linear Evolution Transport Equation: Detection of Pollution Source." *Inverse Problems in Science and Engineering* 20, no. 3 (2012): 401–421.

9 Hamdi A., and Mahfoudi I. "Inverse Source Problem in a One-Dimensional Evolution Linear Transport Equation with Spatially Varying Coefficients: Application to Surface Water Pollution." *Inverse Problems in Science and Engineering* 21, no. 6 (2013): 1007–1031.

10 Permanoor E., Mazaheri M., and Amiri S. "An Analytical Solution for the Advection-Dispersion Equation Inversely in Time for Pollution Source Identification." *Physics and Chemistry of the Earth* (2022): 103255.

11 Biccari U., Song Y., Yuan X., and Zuazua E. "A Two-Stage Numerical Approach for the Sparse Initial Source Identification of a Diffusion-Advection Equation." *Inverse Problems* 39, no. 9 (2023): 095003.

12 Shi C., Cheng H., and Lui K. "The Iterative Generalized Quasi-Boundary Value Method for an Inverse Source Problem of Time Fractional Diffusion-Wave Equation in a Cylinder." *Mathematical Methods in the Applied Sciences* 47, no. 4 (2024): 2006–2029.

13 Bensalah M., and Hassine M. "Inverse Source Problem for a Diffusion Equation Involving the Fractional Spectral Laplacian." *Mathematical Methods in the Applied Sciences* 44, no. 1 (2020): 917–936.

14 Ali M., Aziz S., and Malik S.A. "Inverse Source Problems for a Space-Time Fractional Differential Equation." *Inverse Problems in Science and Engineering* 28, no. 1 (2019): 47–68.

15 Kulbay M., Maussumbekova S., and Mukanova B. *Numerical Solution of Continuation Problem for 3D Steady State Diffusion in Cylindrically Layered Medium*. *Abstract and Applied Analysis*, 2015, Article ID 329052. https://doi.org/10.1155/2015/329052.

16 Nurseitov D.B., and Kasenov S.E. "Numerical Solution of Initial-Boundary Problem for Helmholtz Equation on 'Discretization-Optimization'." *Bulletin of the Karaganda University. Mathematics* 75, no. 3 (2014): 114–120.

17 Rysbaiuly B., and Alpar S.D. "Solution of a Nonlinear Heat Transfer Problem Based on Experimental Data." *Journal of Mathematics, Mechanics and Computer Science* 114, no. 2 (2022): 142–155.

18 Hasanov A. "Simultaneous Determination of Source Terms in Linear Parabolic Problem from the Final Data Overdetermination: Weak Solution Approach." *Journal of Mathematical Analysis and Applications* 330, no. 2 (2007): 766–779.

19 Prilepko A.I., and Tkachenko D.S. "Well-Posedness of the Inverse Source Problem for Parabolic Systems." *Differential Equations* 40, no. 11 (2004): 1619–1626.

20 Johansson T., and Lesnic D. "Determination of a Spacewise Dependent Heat Source." *Journal of Computational and Applied Mathematics* 209 (2007): 66–80.

21 Hasanov A., and Erdem A. "Monotonicity of Input-Output Mappings in Inverse Coefficient and Source Problems for Parabolic Equations." *Journal of Mathematical Analysis and Applications* 335, no. 2 (2007): 1435–1451.

22 Hasanov A. "An Inverse Source Problem with Single Dirichlet Type Measured Output Data for a Linear Parabolic Equation." *Applied Mathematics Letters* 24 (2011): 1269–1273.

23 Hasanov A., and Pektaş B. "A Unified Approach to Identifying an Unknown Spacewise Dependent Source in a Variable Coefficient Parabolic Equation from Final and Integral Overdeterminations." *Applied Numerical Mathematics* 78 (2014): 49–67.

24 Kian Y., Yamamoto M., and Liu Y. "Uniqueness of Inverse Source Problems for General Evolution Equations." *Communications in Contemporary Mathematics* 25, no. 6 (2022). https://doi.org/10.1142/s0219199722500092.

25 Engl H., Scherzer O., and Yamamoto M. "Uniqueness and Stable Determination of Forcing Terms in Linear Partial Differential Equations with Overspecified Boundary Data." *Inverse Problems* 10 (1994): 1253–1276.

26 Yamamoto M. "Stability, Reconstruction Formula and Regularization for an Inverse Source Hyperbolic Problem by a Control Method." *Inverse Problems* 11 (1995): 481–496.

27 Hoffmann B., and Kindermann S. "On the Degree of Ill-Posedness for Linear Problems with Noncompact Operators." *Methods and Applications of Analysis* 17 (2010): 445–462.

28 Hasanov A., Mukanova B. "Fourier Collocation Algorithm for Identifying the Space-Dependent Source in the Advection-Diffusion Equation from Boundary Data Measurements." *Applied Numerical Mathematics* 97 (2015): 1–14.

29 Dahlberg B., and Trubowitz E. "The Inverse Sturm-Liouville Problem III." *Communications on Pure and Applied Mathematics* 37 (1984): 255–267.

30 Vang Z., and Liu J. "Identification of the Pollution Source from One Dimensional Parabolic Equation Models." *Applied Mathematics and Computation* 219 (2012): 3403–3413.

Author Biography:

Kulbay Magira – PhD, Senior Lecturer of Mathematical and Computer Modelling Department, al-Farabi Kazakh National University (Almaty, Kazakhstan, e-mail: makulbay@gmail.com)