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A boundary inverse problem for a second order differential operator

Abstract. In this paper we investigate a boundary inverse problem of a second order differential operator with integral boundary conditions in $L_2(0, b)$, where $b < \infty$. A boundary inverse problem of spectral analysis is the problem of recovering boundary conditions of the operator by its spectrum and some additional data. Usually, as the additional spectral data takes the spectral function of the operator as it occurred in the famous work of I.M. Gelfand and B.M. Levitan. In other cases, as additional data perform spectra of some related operators. We research inverse problem of spectral analysis of second order differential operators with integro–differential boundary conditions. In this case, it is necessary to find from the spectral data not only coefficients of the differential expression, also, we need to find boundary functions of the integro–differential boundary conditions. Coefficient inverse problems are well studied. Therefore in this paper we study the issue of reconstruction of boundary functions. As a main result a uniqueness theorem of the inverse boundary problem in $L_2(0, b)$ was proved.
Keywords: boundary inverse problem, uniqueness theorem, differential operator, integro–differential boundary condition, spectral analysis, spectrum, eigenfunction, associated function, basis, conjugate system, biorthogonal system

Introduction

A boundary inverse problem of spectral analysis is the problem of recovering boundary conditions of the operator by its spectrum and some additional data. Usually, as the additional spectral data takes the spectral function of the operator as it occurred in the famous work of I.M. Gelfand and B.M. Levitan [6]. In other cases, as additional data perform spectra of some related operators. Similar approach can be seen in the works of L.S. Leibenson [10] and V.A. Yurko [20]. In the works of V.A. Marchenko [11] additional spectral data is the scattering data.

Note that differential operators on the interval depending on the type of boundary conditions are divided into operators with local or nonlocal boundary conditions. For example, standard Dirichlet and Neumann boundary conditions refer to the local boundary conditions, while periodic boundary conditions are nonlocal. In the monograph [11] local boundary conditions are called splitting, and nonlocal two–point boundary conditions are called nonseparated boundary conditions. As is known, operators with splitting boundary conditions is much easier to recover from the spectral data. Less developed recovery techniques of differential operators with nonseparated boundary conditions. Reconstruction of second order differential operators with nonseparated boundary conditions can be found in works V.A. Sadovnichii and his students [15, 16, 2, 3].

In this paper we investigate the inverse problem of spectral analysis of second order differential operators with integro–differential boundary conditions. In this case, it is necessary to find from the spectral data not only coefficients of the differential expression, also, we need to find boundary functions of the integro–differential boundary conditions. Coefficient inverse problems are well studied. Therefore in this paper we study the issue of reconstruction of boundary functions.

Now we proceed to accurate formulation of the boundary inverse problem of spectral analysis of the differential operator on the interval. To do this, let us first consider the direct problem of spectral analysis.

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Direct problem

Let $b < \infty$ and in $L_2(0,b)$ there is given the operator $\mathcal{L}$ generated by the second order differential expression

$$l(y) \equiv y''(x) + p(x)y(x),\ \ 0 < x < b$$

with smooth coefficient

$$p \in C[0,b],$$

and boundary conditions

$$U_j(y) \equiv V_j(y) + \sum_{s=0}^{k_j} \int_0^b y^{(s)}(t)\rho_{js}(t)dt = 0,\ \ j = 1,2, \quad (2)$$

where

$$V_j(y) \equiv \alpha_j y^{(k_j)}(0) + \beta_j y^{(k_j)}(b),$$

and α_j, β_j are some numbers, $\rho_{js} \in L_2(0,b)$.

In what follows, we assume that boundary conditions (2) are normed and regular (strongly regular) in A.A. Shkalikov sense (see [2]).

Then we have

Theorem 1. [2] *Eigen– and associated functions of the operator $\mathcal{L}$ with regular (strongly regular) boundary conditions (2) are Riesz basis with brackets (Riesz basis) in $L_2(0,b)$.*

Note, that there exist the set of functions $\{\sigma_i \in L_2(0,b),\ i = 1,2\}$ such that boundary conditions (2) will be equivalent to the conditions

$$U_j(y) \equiv V_j(y) + \int_0^b l(y)\overline{\sigma_j(t)}dt = 0,\ \ j = 1,2,\ (3)$$

as functionals

$$\Phi_j(y'') \equiv \sum_{s=0}^{k_j} \int_0^b y^{(s)}(t)\rho_{js}(t)dt,\ \ j = 1,2$$

are continuous in $L_2(0,b)$.

Hereinafter, functions σ_1, σ_2 will be called boundary functions.

Consider the spectral problem

$$l(y) = \lambda y(x),\ \ 0 < x < b$$

with boundary conditions (3).

Direct problem of spectral analysis (3)–(4) is investigation the geometry of location of eigenvalues and completeness, minimality and basis property of the corresponding system of root functions in the space $L_2(0,b)$. Since if the system of eigen– and associated functions of problem (3)–(4) is Riesz basis with brackets (Riesz basis) in $L_2(0,b)$, then (see [3]) there is a unique biorthogonal system, and the conjugate system is also Riesz basis with brackets (Riesz basis) in $L_2(0,b)$.

Statement of the boundary inverse problem

It needs to find three functions from the spectral data to completely restore the boundary value problem (3)–(4):

$$p \text{ is the coefficient from } (4),$$
$$\sigma_1, \sigma_2 \text{ are boundary functions}.$$

In this paper we study the partial inverse problem. Let the coefficient of the equation (4) is well–known. By the spectral data it needs to restore only boundary functions. It remains to clarify what we understand by the spectral data. So the spectral data of the boundary value problem (3)–(4) is spectra of the following boundary value problems.

First boundary value problem.

$$l(y) = f(x),\ \ 0 < x < b,$$

$$V_1(y) - \int_0^b l(y)\overline{\sigma_1(x)}dx = 0,$$

$$V_2(y) = 0.$$

Second boundary value problem.

$$l(y) = f(x),\ \ 0 < x < b,$$

$$V_j(y) - \int_0^b l(y)\overline{\sigma_j(x)}dx = 0,\ \ j = 1,2.$$

The main result is a theorem of the uniquely reconstruction of boundary functions from the spectra of indicated boundary value problems. More precise formulation of the result is given below.

Necessary formulations and statements

Let $1 \le k \le 2$. Introduce a function $\kappa_k(x,\lambda)$, which satisfies the equation

$$l(\kappa_k) = \lambda\kappa_k(x,\lambda),\ \ 0 < x < b$$

and boundary conditions

$$V_j(\kappa_k) - \lambda\int_0^b \kappa_k(x,\lambda)\overline{\sigma_j(x)}dx = 0,$$

$$j = 1,\dots,k-1,$$

$$V_k(\kappa_k) = \Delta_{k-1}(\lambda),$$

$$V_j(\kappa_k) = 0,\ \ j = k+1,\dots,2,$$

where (5)

$$\Delta_{k-1}(\lambda) = (-1)^{k-1}\Delta_0(\lambda)\ \cdot\ det(\mathrm{E}_{k-1}\Delta_0(\lambda) - \lambda \parallel <\psi_j,\sigma_\nu>; j,\nu = 1,\dots,k-1 \parallel),$$

$$\kappa_k(x,\lambda) = \\ = \ det\begin{pmatrix} \psi_1(x,\lambda) & \dots & \psi_{k-1}(x,\lambda) & \psi_k(x,\lambda) \\ \Delta_0(\lambda) - \lambda <\psi_1,\sigma_1> & \dots & -\lambda <\psi_{k-1},\sigma_1> & -\lambda <\psi_k,\sigma_1> \\ \vdots & \ddots & \vdots & \vdots \\ -\lambda <\psi_1,\sigma_{k-1}> & \dots & \Delta_0(\lambda) - \lambda <\psi_{k-1},\sigma_{k-1}> & -\lambda <\psi_k,\sigma_{k-1}> \end{pmatrix}, \tag{9}$$

E_k is the $k \times k$ unit matrix and $<\cdot,\cdot>$ is the inner product in $\mathrm{L}_2(0,b)$. Here $\{\psi_i\}_{i=1}^2$ is the fundamental system of solutions of the equation $l(\psi) = \lambda\psi$, elements of which satisfy conditions

$$V_j(\psi_k) = \delta_{kj}\Delta_0(\lambda),\ \ k,j = 1,2,$$

$$\Delta_0(\lambda) = det(\parallel V_\nu(y_j); \nu,j = 1,2 \parallel),$$

$$\psi_k(x,\lambda)\ = (-1)^k det\begin{pmatrix} y_1(x,\lambda)\ y_2(x,\lambda) \\ V_1(y_1)\ V_1(y_2) \end{pmatrix},$$

where δ_{kj} is the Kronecker symbol, $\{y_i\}_{i=1}^2$ is the fundamental system of solutions of the equation $l(y) = \lambda y$, elements of which satisfy conditions

$$y_k^{(j-1)}(0) = \delta_{kj},\ \ k,j = 1,2.$$

To check the relation (9), it is sufficiently to show that the right–hand of (9) satisfies all conditions, which satisfies the function $\kappa_k(x,\lambda)$.

If $k = 2$ then conditions (8) absent. If $k = 1$ then conditions (6) absent.

Corollary 1. *The functions $y_k(x,\lambda), \psi_k(x,\lambda), \kappa_k(x,\lambda)$ are entire functions respect to λ for all $k = 1,2$.*

So, in what follows we admit that the following condition is valid.

A) Spectra of any two boundary value problems do not intersect, that is

$$|\Delta_k(\lambda)|^2 + |\Delta_j(\lambda)|^2 \neq 0$$

for all $\lambda \in \mathbb{C}$ and $k \neq j$.

Syne a function identically not equal to zero have either a finite number or a countable number of zeros without finite limit points, let us denote by

$$|\lambda_1^{(k)}| \le |\lambda_2^{(k)}| \le \dots,$$

zeros of the function $\Delta_k(\lambda)$. The entire function $\Delta_0(\lambda)$ equal to 1 at $\lambda = 0$, hence satisfies this condition. Zeros of an entire function can be have finite multiplicity. Denote by $\theta_m^{(k)}$ the multiplicity of eigenvalue $\lambda_m^{(k)}$, i.e.

$$\Delta_k^{(\nu)}(\lambda_m^{(k)}) = 0 \quad as \quad \nu = 0,1,\dots,\theta_m^{(k)} - 1, \qquad (10)$$
$$\Delta_k^{(\theta_m^{(k)})}(\lambda_m^{(k)}) \neq 0.$$

Let us introduce below two systems of functions. Let $1 \leq k \leq 2$. Then for every k we put

$$u_{m,k}(x) = \kappa_k(x, \lambda_m^{(k)}), \ \ m \geq 1,$$
$$u_{m+1,k}(x) = \frac{1}{1!}\frac{\partial}{\partial\lambda}\kappa_k(x,\lambda)|_{\lambda=\lambda_m^{(k)}}, \ \ m \geq 1,$$
$$\dots\dots\dots, \qquad (11)$$
$$u_{m+\theta_m^{(k)}-1,k}(x) = \frac{1}{(\theta_m^{(k)}-1)!}\frac{\partial^{\theta_m^{(k)}-1}}{\partial\lambda^{\theta_m^{(k)}-1}}\kappa_k(x,\lambda)|_{\lambda=\lambda_m^{(k)}}, \ \ m \geq 1.$$

From (9) follows, that $\kappa_k(x,\lambda)$ depends only on $\sigma_1,\dots,\sigma_{k-1}$. Thus, if boundary functions $\sigma_1,\dots,\sigma_{k-1}$ and zeros of $\Delta_k(\lambda)$ are well–known then the system (11) is completely defined.

Proposition 1. *For a fixed admissible* k *and* m *the system of functions (11) is a chain of eigenfunctions and associated functions corresponding to the eigenvalue* $\lambda_m^{(k)}$, *i.e.* $u_{m,k}(x)$ *eigenfunction of* k *th boundary problem and* $u_{m+i,k}(x)$ *associated functions of the same problem for all* $i = 1,\dots,\theta_m^{(k)} - 1$.

Proof. We note, that the function $\kappa_k(x,\lambda)$ is a solution of the equation

$$l(\kappa_k(\cdot,\lambda)) = \lambda\kappa_k(x,\lambda), \quad 0 < x < b$$

and satisfies boundary conditions

$$V_j(\kappa_k) - \int_0^b l(\kappa_k)\overline{\sigma_j(x)}dx = 0, \ \ j = 1,\dots,k-1,$$
$$V_k(\kappa_k) - \int_0^b l(\kappa_k)\overline{\sigma_k(x)}dx = \Delta_k(\lambda),$$
$$V_j(\kappa_k) = 0, \ \ j = k+1,\dots,n.$$

By using the relations (10) from (12), we get Proposition 1. For example, let us check Proposition 1 for $u_{m,k}(x)$. In the relation (12) substitute $\lambda = \lambda_m^{(k)}$, and take into account first relation from (10). Then

$$l(u_{m,k}) = \lambda_m^{(k)} u_{m,k}, \quad 0 < x < b$$

$$V_j(u_{m,k}) - \int_0^b l(u_{m,k})\overline{\sigma_j(x)}dx = 0, \ \ j = 1,\dots,k,$$

$$V_j(u_{m,k}) = 0, \ \ j = k+1,\dots,2.$$

Other relations for $u_{m+j,k}(x)$ verify similarly. Only needs to differentiate by λ required number times and instead of λ substitute $\lambda_m^{(k)}$.

Proposition 1 is proved.

Proposition 2. *The solution of the inhomogeneous equation*

$$l(y) = \lambda y(x) + f(x), \ \ 0 < x < b$$

with the boundary conditions

$$V_j(y) - \int_0^b l(y)\overline{\sigma_\nu(x)}dx = 0, \ \ \nu = 1,\dots,k,$$
$$V_j(y) = 0, \ \ \nu = k+1,\dots,2$$

given by the formula

$$y(x,\lambda) = \int_0^b G_k(x,t,\lambda)f(t)dt, \qquad (12)$$

where

$$G_k(x,t,\lambda) = (-1)^k\left(\prod_{s=1}^{k}\Delta_s(\lambda)\right)^{-1} \times$$

$$\times\begin{vmatrix} \kappa_1(x,\lambda) & \kappa_2(x,\lambda) & \dots & \kappa_k(x,\lambda) & G_0(x,t,\lambda) \\ \Delta_1(\lambda) & 0 & \dots & 0 & U_1(G_0) \\ U_2(\kappa_1) & \Delta_2(\lambda) & \dots & 0 & U_2(G_0) \\ \dots & \dots & \ddots & \dots & \dots \\ U_k(\kappa_1) & U_k(\kappa_2) & \dots & \Delta_k(\lambda) & U_k(G_0) \end{vmatrix},$$

$G_0(x,t,\lambda)$ is the Green function of the boundary value problem

$$l(y)=\lambda y(x),\ \ 0<x<b$$

with the boundary conditions

$$V_j(y)=0,\ \ j=1,2.$$

Here $U_1(y),\dots,U_k(y)$ are forms of the boundary conditions (13).

Proof. Proposition 2 proves by checking the equation and the boundary conditions.

Corollary 2. *From Proposition 2 follows that the Green function $G_k(x,t,\lambda)$ has the form*

$$G_k(x,t,\lambda)=\sum_{i=1}^{2}(-1)^{i+1}\kappa_i(x,\lambda)M_i(t,\lambda)+G_0(x,t,\lambda),$$

where the determinant M_i is taken by substitution the first row of determinant G_k with

$$(0,\dots,0,1,0,\dots,0),$$

where unit on the ith place.

Let us calculate the reduce of the Green function $G_k(x,t,\lambda)$ at the singular point $\lambda_m^{(k)}$. Indeed, it is related to the kernel of the projection onto the root subspace of the corresponding eigenvalue $\lambda_m^{(k)}$. The equality

$$res_{\lambda_m^{(k)}}G_k(x,t,\lambda)=res_{\lambda_m^{(k)}}(-1)^{k+1}\kappa_k(x,\lambda)M_k(t,\lambda),\tag{15}$$

holds, since by the condition A) spectra of considered boundary value problems are not intersect and, therefore the remaining terms have zero residues at $\lambda_m^{(k)}$. Indeed, the function $G_0(x,t,\lambda)$ meromorphic respect to λ but does not have a pole at $\lambda_m^{(k)}$. Similarly, we can prove that the meromorphic function $M_i(t,\lambda)$ for $i<k$ regular at $\lambda_m^{(k)}$.

By using the equality (11), as a result from (15) we get

$$res_{\lambda_m^{(k)}}G_k(x,t,\lambda)=\sum_{j=0}^{\theta_m^{(k)}-1}u_{m+j,k}(x)h_{m+\theta_m^{(k)}-1-j,k}(t),$$

where

$$h_{m+\theta_m^{(k)}-1-j,k}(t)=\frac{1}{(\theta_m^{(k)}-1-j)!}\lim_{\lambda\to\lambda_m^{(k)}}\frac{\partial^{\theta_m^{(k)}-1-j}}{\partial\lambda^{\theta_m^{(k)}-1-j}}\left[(\lambda-\lambda_m^{(k)})^{\theta_m^{(k)}-1}M_k(t,\lambda)\right].\tag{16}$$

Proposition 3. *The system of functions $\{h_{m+i,k},\ i=0,1,\dots,\theta_m^{(k)}-1\}$ is conjugate system to the system $\{u_{m+j,k},\ j=0,1,\dots,\theta_m^{(k)}-1\}$ in $L_2(0,b)$, i.e.*

$$<u_{m+j,k},h_{m+\theta_m^{(k)}-1-s,k}>=\delta_{js},$$

where δ_{js} is the Kronecker symbol.

Proof. The proof of Proposition 3 follows from M. Riesz's theorem of projectors onto the root subspace, which are calculated as residue of resolvent. In our case, instead of resolvent we have the function $G_k(x,t,\lambda)$ corresponding to the boundary value problem.

Remark. In Proposition 3 biorthogonality proved for root functions from the same root subspace. If the root subspaces corresponding to the different eigenvalues then orthogonality of such root subspaces is well–known.

Main result

The following theorem is the main result of the work.

Theorem 2. *Let us given all eigenvalues $\{\lambda_m^{(2)},\ m\geq 1\}$ of the boundary problem (3)–(4). Additionally, assume there are given spectra of else one boundary problem $\{\lambda_m^{(1)},\ m\geq 1\}$, which are ensue from the initial problem (3)–(4) by gradual zeroing integral perturbations of the boundary conditions. Then boundary functions σ_1,σ_2 from (3) uniquely recover.*

Proof. Suggest a reconstruction algorithm of the boundary functions σ_1,σ_2.

At first, consider the case when eigenvalues $\{\lambda_m^{(k)},\ m \geq 1\}$ have a simple multiplicity for all $1 \leq k \leq 2$.

First step. Reconstruction of σ_1 by the spectrum of the first boundary value problem from $L_2(0,b)$. Let us given the sequence of eigenvalues $\{\lambda_m^{(1)},\ m \geq 1\}$ of the first boundary value problem. Construct the function $\kappa_1(x,\lambda)$ as a solution of the Cauchy problem

$$l(\kappa_1) = \lambda\kappa_1(x,\lambda),\ \ 0 < x < b$$

with the condition at zero

$$V_1(\kappa_1) = \Delta_0(\lambda), \qquad V_2(\kappa_1) = 0.$$

Such solution exists for all complex λ, in particular, for $\lambda = \lambda_m^{(1)}$. Hence, there constructs the system of root functions

$$\{u_{m,1}(x) = \kappa_1(x,\lambda_m^{(1)}), \qquad m \geq 1\}.$$

From works A.A. Shkalikov [2] and N.K. Bari [3] follow, that this system is Riesz basis with brackets (Riesz basis) in $L_2(0,b)$, and has a unique conjugate system, which is also Riesz basis with brackets (Riesz basis) in $L_2(0,b)$. Then the Fourier coefficients of the boundary function σ_1 by the system $\{h_{m,1},\ m \geq 1\}$ have the form

$$< u_{m,1}, \sigma_1 >= \frac{\Delta_0(\lambda_m^{(1)})}{\lambda_m^{(1)}},$$

as $\Delta_1(\lambda_m^{(1)}) = 0$ for all $m \geq 1$ and $\Delta_1(\lambda) = \Delta_0(\lambda) - \lambda\int_0^b \kappa_1(x,\lambda)\overline{\sigma_1(x)}dx$. Since the system $\{h_{m,1},\ m \geq 1\}$ is basis, we can construct the function σ_1 from $L_2(0,b)$, i.e.

$$\sigma_1(x) = \sum_{m=1}^{\infty} \frac{\Delta_0(\lambda_m^{(1)})}{\lambda_m^{(1)}} h_{m,1}(x).$$

Thus, one of the boundary functions is reconstructed.

Second step. Reconstruction of σ_2 by the spectrum of the second boundary value problem and by known σ_1 from $L_2(0,b)$. Let us given the sequence of eigenvalues $\{\lambda_m^{(2)},\ m \geq 1\}$ of the second boundary value problem. Construct the function $\kappa_2(x,\lambda)$ as a solution of the following Cauchy problem

$$l(\kappa_2) = \lambda\kappa_2(x,\lambda),\ \ 0 < x < b$$

with conditions

$$V_1(\kappa_2) - \lambda\int_0^b \kappa_2(x,\lambda)\overline{\sigma_1(x)}dx = 0,$$
$$V_2(\kappa_2) = \Delta_1(\lambda).$$

Hence, constructs the system of eigen– and associated functions

$$\{u_{m,2}(x) = \kappa_2(x,\lambda_m^{(2)}), \qquad m \geq 1\}.$$

Then the Fourier coefficients of the boundary function σ_2 by the system $\{h_{m,2},\ m \geq 1\}$ have the form

$$< u_{m,2}, \sigma_2 >= \frac{\Delta_0(\lambda_m^{(2)})}{\lambda_m^{(2)}},$$

as $\Delta_2(\lambda_m^{(2)}) = 0$ for all $m \geq 1$ and $\Delta_2(\lambda) = \Delta_0(\lambda) - \lambda\int_0^b \kappa_2(x,\lambda)\overline{\sigma_2(x)}dx$. Since the system $\{h_{m,2},\ m \geq 1\}$ is basis, we can construct the function σ_2 from $L_2(0,b)$

$$\sigma_2(x) = \sum_{m=1}^{\infty} \frac{\Delta_0(\lambda_m^{(2)})}{\lambda_m^{(2)}} h_{m,2}(x).$$

Thus, the second boundary function is reconstructed.

In the case, when eigenvalues are non simple (indeed, formulas (17), (18) slightly become more complicated (see (11))), by the analogous discussions (except, may be, with technical difficulties), we get required assertion.

Theorem 2 is proved.

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